CULLEN COLLEGE OF ENGINEERING

UNIVERSITY OF HOUSTON

HOUSTON, TEXAS

SEMI-ANNUAL PROGRESS REPORT

GRANT NGR-44-005-039

GENERAL MULTIPLEXING THEORY

submitted to the

National Aeronautics and Space Administration

by

Electrical Engineering Department
University of Houston
Houston, Texas   77004

N 67 32579

PART I

GENERAL MULTIPLEXING THEORY

by

R.D. Shelton

I. INTRODUCTION

A. Technical Background

Multiplexing is simply the process by which two or more message waveforms are combined into a single waveform for transmission over a single channel. The only essential requirement for a multiplexing system is that the receiver must be able to recover all of the message waveforms from the single received waveform or, in other words, the multiplexer transformation must be <u>reversible</u>.

The block diagram of Figure 1 is useful for visualization of the multiplexing process. The symbols defined in the figure will be used throughout this paper. There are N message channels whose waveforms are denoted by $m_i(t)$. The single waveform out of the multiplexer is $f_m(t)$. This waveform is transmitted through a single channel which here is assumed to include transmitter carrier modulation and the corresponding receiver carrier demodulation. The received version of $f_m(t)$ is denoted by $f_m^*(t)$ to account for differences due to noise and distortion caused by imperfection in the channel. In general the asterisk is used to denote a receiver estimate of a transmitted quantity. Thus the demultiplexing process is said to produce outputs $m_i^*(t)$ which should be as close as possible to the original message waveforms.

There are two conventional methods of multiplexing, frequency division multiplexing (FDM) and time division multiplexing (TDM). These two methods have been used so predominantly that many people had the impression that they were the only possible multiplexing methods. However, from time to time other multiplexing methods were used, and in 1952 Zadeh and Miller wrote a paper demonstrating, in effect, that there were infinitely many ways of multiplexing signals in a reversible manner. They showed that multiplexing systems can be based on any set of orthogonal waveforms, of which infinite varieties exist. Later a number of papers were written, each describing a new type of multiplexing system and claiming that each of these systems was superior in some sense to conventional systems. All of these systems will be discussed in the next chapter, but here the point is that the publication of these papers created the requirement for a general analysis:

1. To determine a general model which could describe all conceivable multiplexing systems, and

2. To determine by use of this model which types of multiplexing systems are optimum for certain important channels.
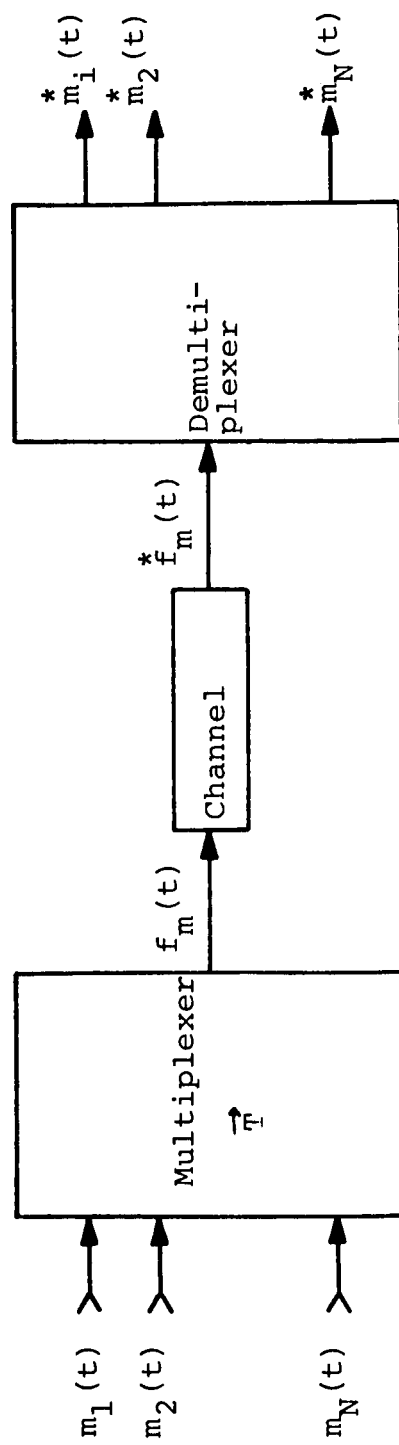
Figure 1. Basic block diagram for a general multiplexing system

B. Project Background

It is desirable to include here some of the basic background information on the project. Included will be a short description of other publications resulting from the grant.

Grant NGR-44-005-039 was given by the National Aeronautics and Space Administration to the University of Houston on May 17, 1966. The principal investigator was R.D. Shelton. The grant was in the amount of $51,356 and was to run for one year. A renewal was received on February 26, 1967 to support the project for a second year with the amount of $52,458.

The title of the grant was "Advancement of the General Theory of Multiplexing with Applications to Space Communications." The grant consisted of three major tasks: 1) development of new methods of signal multiplexing, 2) performance comparison of new multiplexing systems with conventional systems and 3) application of the best of these new techniques to the design of a modulation and multiplexing system for the Apollo Applications Program.

Table 1 lists technical personnel active in performing grant tasks during its first year. In addition to those listed a full-time secretary and a half-time draftsman supported the project. The publications resulting from these studies are listed in chronological order in Table 2. Publications in progress in June 1967 are listed in Table 3.

NAME	TITLE	PERCENT TIME	MONTHS ON PROJECT (To June 1, 1967)	SUBJECT OF WORK
Dr. H.S. Hayre	Professor	25%	12	Review of all technical work-consultation.
Dr. E.L. Michaels	Professor	25%	4	Picture transmission for AAP.
Dr. J.D. Bargainer	Assistant Professor	25%	4	Digital Processing for AAP Communications System
R.D. Shelton	Principal Investigator	50%	12	Project Manager to March 1967. General Study of multiplexing.
T. Williams	Research Associate	100%	9	Assistant Project Manager, study of Feasibility of Orthomux systems.
S. Riter	Research Associate	100%	8	Project Manager after March 1967. System Design for MIMCOM.
S. Sloan	Computer Programmer	100%	4	Programming service, Digital simulations.
L. Puigjaner	Graduate Assistant	50%		Application of Analytic ...als to Multiplexing ...ysis.
P. Weinreb	Graduate Assistant	50%		...sted Puigjaner ...of grant--see

Table 1. Technical personnel active during next page for continuation.

NAME	TITLE	PERCENT TIME	MONTHS ON PROJECT (To June 1,1967)	SUBJECT OF WORK
S.Z.H. Taqvi	Graduate Assistant	50%	4	Phaselock loop optimization for MIMCOM
C. Osborn	Student Assistant	25%	6	Analog Simulations
W. Trainor	Student Assistant	25%	6	Analog Simulations
M. Smither	Student Assistant	25%	3	Telemetry date compression for MIMCOM
W.L. Hon	Technician	100%	6	Developed and tested carrier hatching loop for MIMCOM
J. Froeschner	Technician	100%	3	Assisted Analog Simulations
S. Vaharami	Student Assistant	25%	4	Assisted Analog Simulations
S. Wade	Technical Writer	50%	12	Writing Service

Table 1.(continuation of)Technical personnel active during first year of grant.

Date of Presentation	Title	Place of Presentation	Author
Sept. 1966	"Analytic Signals and Zero Locus in Multiplexing Systems"	M.S. Thesis	L. Puigjaner
June 1967	"An Optimum Multiplexing System for Space Communications"	M.S. Thesis	S. Riter
April 1967	Four Senior Papers		W.L. Hon S. Sloan C. Osborn and W. Trainor J. Froeschner
April 1967	"Orthogonal Multiplexing Systems Based on Easily-Generated Waveforms"	SWIEEECO Dallas	T. Williams and R.D. Shelton
June 1967	"A Study of Optimum Multiplexing Systems"	Ph.D. Dissertation	R.D. Shelton
June 1967	"Implementation of Optimum Multiplexing Systems"	Ph.D. Dissertation	T. Williams
June 1967	"Analog and Digital Computer Simulations of Multiplex Systems Performance"	IEEE International Communications Conference, Minneapolis	S. Riter T. Williams R.D. Shelton
June 1967	"Communications Systems for Manned Interplanetary Explorations"	1967 Symposiun of the American Astronautical Society, Huntsville, Alabama	R. Riter and R.D. Shelton

Table 2. Publications Resulting from grant to June 1967.

Planned Date of Presentation	Tentative Title	Place of Presentation	Author
July 1967	"Realization of Optimum Multiplexing Systems"	Ph.D. Dissertation	T. Williams
July 1967	"Signal Design and Coding for a Manned Interplanetary Mission Communication System"	M.S. Thesis	B. Batson
October 1967	"T.V. Systems for Manned Interplanetary Missions"	Eastcon, Washington, D.C.	E.L. Michaels
Feb. 1968	"Data Compression for Manned Interplanetary Mission Communications"	M.S. Thesis	M.A. Smither
Feb. 1968	"Crosstalk in Orthomux Systems"	M.S. Thesis	R.A. Van Cleave
June 1968	"High-Gain Spacecraft Antennas"	Ph.D. Dissertation	I.D. Tripathi
June 1968	"Coding for MIMCOM"	Ph.D. Dissertation	C.Q. Ho
June 1968	"Modulation Optimization for Deep-Space Communications Systems"	Ph.D. Dissertation	S. Riter
June 1968	"Carrier Synchronization Circuits"	M.S. Thesis	W.L. Hon
June 1968	"Theory of Delay Lock Loops"	Ph.D. Dissertation	S.Z.H. Taqvi

Table 3. Publications in progress in June 1967.

The next chapter provides a review of conventional multiplexing systems. A review is also made of the literature on new types of multiplexing systems.

Chapter III is a general study of possible multiplexing systems. A model is derived which describes many interesting types of multiplexing systems. It is shown that each of the new multiplexing systems described in the literature fits into this model. The model may be used to develop a wide variety of new multiplexing systems. More importantly, it permits a rather general analysis of multiplexing system optimality.

In Chapter IV a study is made using the model of Chapter III of the question of which multiplexing system is best. For a channel which disturbs the signal $f_m(t)$ only by the addition of independent white Gaussian noise, it is shown that all multiplexing systems of a very large class perform equally well. Thus the only important criterion of optimality for such a channel is that of equipment simplicity. This can be shown to depend primarily on the ease of generating a set of orthonormal waveforms as the impulse response of a set of linear time-invariant filters. It can be shown that a second way of achieving hardware simplicity is to use multiplexing systems based on binary waveforms. Detailed system analysis and design of these two classes of multiplexing systems are presented by Williams (1967).

For other channels the type of optimum multiplexing system can be more objectively determined. For a channel which band-limits (in frequency) the signal $f_m(t)$, it is shown that a multiplexing system based on prolate spheroidal signals is optimum. In any case ordinary frequency division multiplexing is shown in Chapter V to perform almost as well and is much more practical to implement.

For a channel which has a peak signal amplitude limitation, binary waveforms are shown to be optimum. Thus the selection of optimum systems becomes a problem of coding theory.

When the channel imposes a combination of constraints, the question of multiplexing optimality is much more complicated. It is necessary to be fairly specific about the constraints in order to arrive at the optimum system. In the final section of the Chapter IV an important example is worked out.

In Chapter V performance comparisons are made between the optimum multiplexing systems derived in Chapter IV and the conventional systems presently used for such channels.

II. REVIEW OF MULTIPLEXING SYSTEMS

A. Introduction

The purpose of this chapter is twofold. First a review will be make of conventional multiplexing systems. Detailed performance calculations will not be made, as they are readily available in the literature. A review will also be made of the papers which propose new multiplexing systems. The discussion is limited to multiplexing systems which use a single channel. Systems which exploit multiple waveform transmission capabilities of certain physical channels, such as use of two polarization senses on a radio link for two message channels, are not considered.

There are two conventional kinds of multiplexing, frequency division multiplexing (FDM) and time division multiplexing (TDM). There is no book devoted to a study of multiplexing systems although several books, Black (1953), Nichols and Rauch (1956), Stiltz (1961), and Downing (1964) contain a chapter or two on conventional multiplexing systems. Most of these textbook treatments are based primarily on a single paper by Landon (1948), although the more recent books contain additional topics based on more modern papers.

B. Frequency Division Multiplexing

A general block diagram of a frequency division multiplexing system is presented in Figure 2. As usual, any carrier modulation and demodulation is included in the channel block. The messages modulate a set of sinusoidal subcarriers. Any type of modulation method may be used. The frequencies of the subcarriers are chosen so that, even after modulation, the spectra do not overlap when the modulator outputs are summed. Therefore, the individual spectrum resulting from modulation of a subcarrier by a specific message waveforms can be recovered by frequency filtering. Demodulation of the subcarrier then recovers the message waveform.

The most common classification scheme for frequency division multiplexing systems is based on the type of subcarrier and carrier modulation. If the subcarriers are amplitude modulated and the carrier is frequency modulated, the multiplexing system is called an AM/FM system. Since there are about five common analog modulation methods, there are twenty-five combinations. So-called double multiplexing systems are sometimes used, where one or more of the message waveforms is the output of an FDM (or TDM) multiplexing system. Thus, one finds
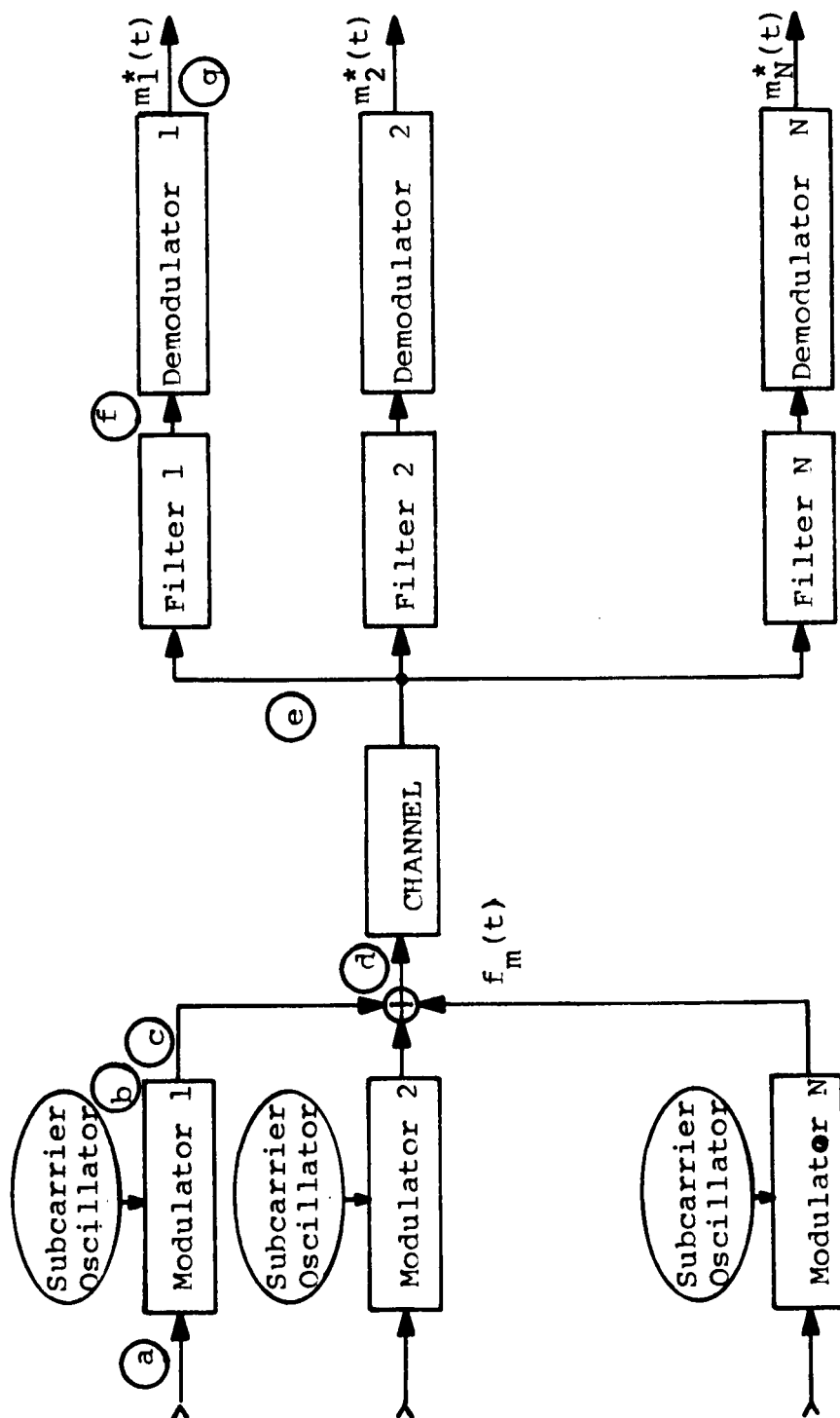
Figure 2. Frequency division multiplex transmitter and receiver.

combinations like PM/AM/FM, which would result if the
message waveforms at the input of the previous example
were actually a sum of phase modulated sub-carriers.
The book by Nichols and Rauch (1956) contains tabulations
of the noise performance of many of the possible combin-
nations. In the same book there are also some elementary
calculations of the crosstalk between message waveforms
due to channel imperfections. Other sources for noise
and crosstalk performance of FDM are Bennett et al (1955),
Florman and Tary (1960), and Nicholas (1954). The problem
of maximization of peak-to-average ratios for optimum
noise performance is considered by Anderson et al (1961),
Brock and McCarty (1955) and is summarized by Downing (1964).
These are the key sources for theoretical results on FDM
systems. There are many papers available on design and
test of specific systems. A good source for such hardware
considerations is the bibliography by Filipowsky and
Bickford (1965).

C. Time Division Multiplexing

A general block diagram of a time division multi-plexing system is presented in Figure 3. The basic prin-ciple of operation of the system is time sharing of the channel by the message channels. This is shown schema-tically by the rotary switch or commutator in the figure. The block following the commutator represents any process-ing that is done on the sample pulses before they are sent to the channel (or carrier modulator) for transmission. The basic output of the commutator is a sequence of samples of the message channels containing the sample values as the amplitudes of the sample pulses. If a sample pulse is simply shaped by the sample processor for improved band-width performance, the system is called a pulse amplitude modulation system (PAM). However, the information contained in the sample amplitude may be modulated onto a pulse in any other way as long as the basic principle of time separation of the samples is maintained. Thus the sample processor may modulate the width or duration of a pulse with the sample value (PWM or PDM). Pulse frequency modulation is used for single channel systems but is not used for TDM systems because it violates the principle of non-lapping time slots for the sample values. However, pulse position modulation(PPM) can be used for TDM systems since the position of the pulses can be restricted so that no two pulses can ever overlap. The most advanced form of sample processing is called pulse code modulation(PCM). In this system the sample values are converted into a binary number , by conversion of the voltage amplitude into its binary equivalent, for instance. The binary number is then used to alternate the polarity of a sequence of pulses, all of which are confined to the time slot allocated to the single sample value.

The receiver must demodulate the received pulses to recover the original sample amplitudes. These sample amplitudes are directed to message channel outputs by another commutator (or decommutator) which must be in time synchronism with the transmitted commutator. The original message waveforms can be recovered by proper interpolation, if the sampling rate is high enough.

Time division multiplexing systems are also clas-sified according to the type of modulation of the basic wave-form, which is a pulse instead of the sinusoid used for FDM. If phase modulation is the carrier modulation process, some possible systems are PAM/PM, PDM/PM PDM/PM, and PCM/PM. Crosstalk in TDM systems usually occurs because of limited
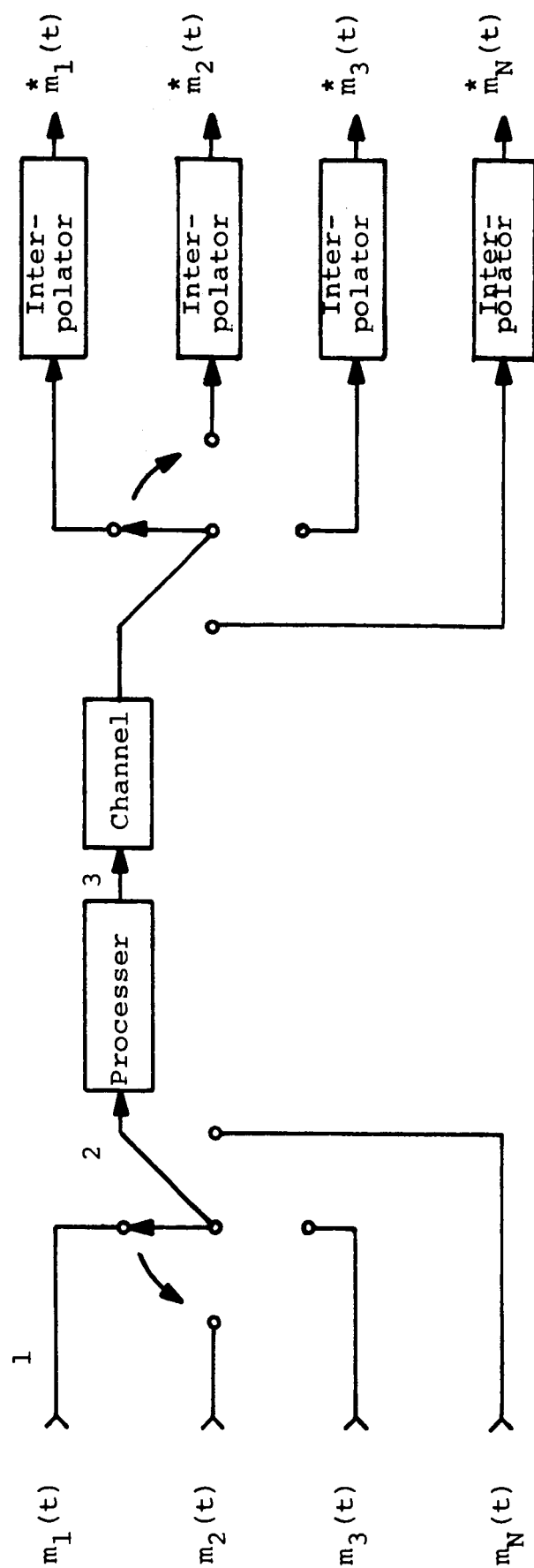
Figure 3   Time division multiplex transmitter and receiver

bandwidth or phase distortion in the channel, although nonlinearities can cause crosstalk as well. The book by Nichols and Rauch (1956) also contains tabulations of the noise and corsstalk performance of most of the common TDM systems. Another good source is the early paper by Bennet (1941). More recent papers are those by Marcatili (1961) and Moskowitz et al (1950). Another good text for TDM is Rowe (1965).

D. Other Multiplexing Systems

Time and frequency multiplexing systems are not the only methods possible for combination of several message waveforms into a single function of time for transmission over a single channel. In 1951 Marchand and Holloway proposed a systematic method for development of other types of multiplexing systems by the use of general orthogonal functions. This method arose because it was observed that separability of the message channels in time and frequency multiplexing at the receiver was based on the orthogonality of nonoverlapping time pulses and sine waves of different frequencies respectively. Since many other functions have the property of orthogonality, it became clear that many other types of multiplexing systems were possible. A paper by Zadeh and Miller (1952) and another paper by Marchand (1953) represent the next contributions to the speculations about such system possibilities.

The first detailed system design and development of multiplexing systems based on other orthogonal functions was made by Ballard in a series of papers (1962a), (1962b), (1962c) and (1963). In the first of these papers the word "orthomux" was coined to describe multiplexing systems in which the message waveforms are linearly multiplied by the orthogonal functions in the transmitter and are recovered by correlation in the receiver. The general block diagram of an orthomux transmitter and receiver is shown in Figure 4 . The orthogonal waveforms are normalized so that:

$$\int_0^T 0_n(t)\,0_m(t)\,dt = \delta_{nm}$$

where δ_{nm} is the Kronecker delta, which is one when the subscripts are the same and zero otherwise.
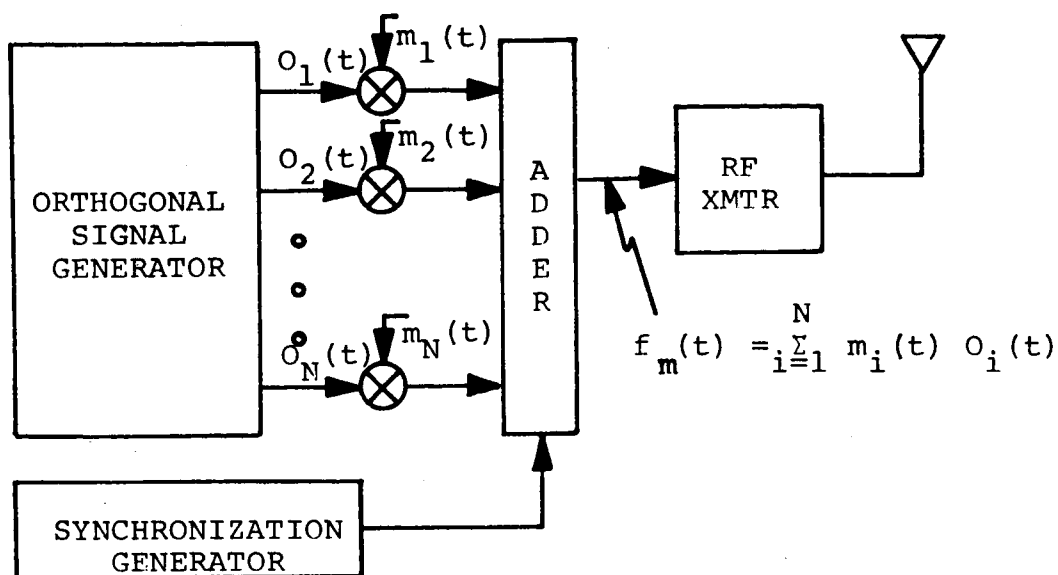
a) Transmitter Block Diagram
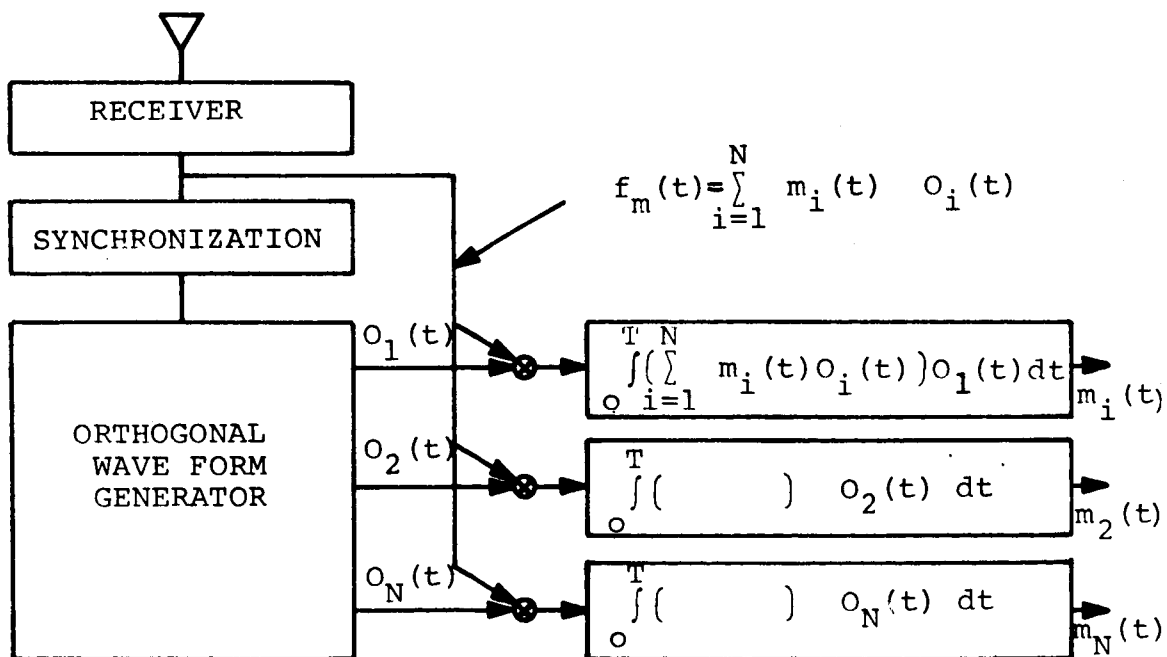
b) Receiver Block Diagram

Figure . Orthomux System a) Transmitter b) Receiver

The output of the multiplex system is the waveform:

$$f_m(t) = \sum_{i=1}^{N} m_i(t) 0_i(t) \qquad t\varepsilon(0,T)$$

It is necessary to assume that the orthogonality interval (0,T) is short enough so that the message waveforms can vary little during the interval, otherwise the terms in the series would no longer be orthogonal to one another. At the end of the interval the orthogonal waveforms are repeated, so that a continuous sequence of messages is sent. It is convenient to then redefine the time origin so that the next interval is also written (0,T). The received waveform is also $f_m(t)$ if the channel is perfect. The jth message waveform can then be recovered by multiplying the received waveform by $0_j(t)$ and integrating over the orthogonality interval:

$$\int_0^T 0_j(t) f_m(t) dt = \int_0^T 0_j(t) \sum_{i=1}^{N} m_i(t) 0_i(t) dt = \sum_{i=1}^{N} m_i \int_0^T 0_j(t) 0_i(t) dt = m_j(t)$$

The assumption that $m_i(t)$ is essentially constant over the orthogonality interval was used in removing it from under the integral sign. It is seen the orthomux system is sucessful in recovering the original message waveform, as long as the receiver is in time synchronization with the transmitter.

Double sideband modulation FDM and pulse amplitude modulation TDM are members of the orthomux class because the message waveform are multiplied by orthogonal sinusoids and time pulses respectively in these systems. Other FDM and TDM systems are essentially orthomux systems except that instead of modulating the basic orthogonal function by multiplication, some other form of modulation is used which does not disturb the orthogonality of the $0_n(t)$. Such systems are called "non-multiplicative orthomux systems" in this dissertation.

Many sets of orthogonal functions are available for use in an orthomux system. Ballard has designed systems using Legendre polynomials and orthogonal binary (Rademacher) functions. Karp and Higuchi (1963) analyzed modified Hermite polynomials. Judge (1962) analyzed another set of binary functions. Many other interesting functions are known.The publication of these papers led to the questions of which of the possible orthomux systems was optimum for specific channels and how superior was its performance to conventional systems.

Although the orthomux model is capable of producing
an infinite variety of new multiplexing systems, it is not
capable of describing all possible multiplexing systems.
A simple multiplexing system for which no orthomux model can
be derived is called amplitude division multiplexing or ADM.
In this system the amplitude of a single waveform is deter-
mined by the value of all the input messages. An example
would be a system in which the messages on two binary inputs
determine one of four possible constant output voltages. It
is clear the input messages can be recovered from the out-
put level so that a valid multiplexing system exists.

Another multiplexing system for which no orthomux
system can be derived is described by Titsworth (1963).
The existence of these multiplexing systems which do not
fit into the framework of the orthomux model raises another
question: Does there exist a model which will describe all
conceivalbe multiplexing systems? It would be very desirable
if such a model could be found as it would allow a very
general optimization to be made multiplexing systems.
These questions are considered in the following chapter.

III. GENERALIZED MULTIPLEXING SYSTEMS

A. Background

The purpose of this chapter is to attempt to develop a model which will describe all possbile multiplexing systems. The most general diagram of a multiplexing system is that shown in Figure 1. One classification for such systems is based on the type of detection process used. Although other possibilities exist, in practice only two types of detectors are used. The first will be called a point or memoryless detector. It bases its decision of which of the input messages was transmitted on a single sample of the received waveform $\hat{f}_m(t)$. The second type of detector bases its decision on the values of $\hat{f}_m(t)$ over an interval of time; consequently it will be called an interval detector. This second type of detector is much more important because it has superior noise performance.

The superiority of interval detectors leads to the consideration of multiplexing systems which are transformations between a vector of input numbers.

$$\vec{m}(nT) = \left[m_1(nT), \ m_2(nT), \cdots, \ m_N(nT) \right] \quad , \ n = 0, \pm 1, \pm 2 \cdots.$$

and a waveform $f_m(t)$ defined for an interval from $t = nT$ to $t = (n + 1)T$.

$$\vec{T}\left[\vec{m}(nT) \right] = f_m(t) \qquad \text{for } t\epsilon (nT, \ (n+1)T)$$

Such a viewpoint is strictly correct if the input messages are binary (or M-ary) waveforms that have the same bit intervals. It is also essentially correct if the input messages are continuously varying (analog) waveforms that are bandlimited. At the present time, however, it will be helpful to visualize the input message waveforms as each having an infinite number of levels, and all of them having changing levels (if they change) at multiples of T seconds from the time origin. For each of the intervals a vector $\vec{m}(nT)$ represents the input messages, and the multiplexer produces a single waveform $f_m(t)$. A multiplexing system could easily be constructed which would "mix-up" the intervals. That is, the waveform $f_m(t)$ in one interval could depend on messages in other intervals. This case will not be considered here. It is said that the remaining class of multiplexing systems are real-time systems.

It is clear that the only essential requirement for the multiplexing process to be reversible is that different

input message vectors must lead to different waveforms $f_m(t)$.

$$\vec{m}_1 \neq \vec{m}_2 \Rightarrow f_{m_1}(t) \quad \text{for all } t \epsilon (0,T) \tag{1}$$

Therefore if the ith channel has k_i different levels (or messages) then the total number of different messages is:

$$K = \prod_{i=1}^{N} k_i$$

and K different waveforms $f_m(t)$ are required. The following derivation leads to a block diagram model for the multiplexing system described in general terms above. The derivation is similar to that used by Wozencraft and Jacobs (1965) for the single channel case. First, however, a primitive model of a multiplexing system will be described.

It is easy to see that one model which will describe any such multiplexing system is that shown in Figure 5. When the input message vector is m_i, the vector transformation produces an output of one for c_i and zero for all the other coefficients:

$$\vec{C}\left[\vec{m}_i\right] = \vec{c}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \ldots \delta_{ik}) \tag{2}$$

This model is of some value for visualization of the multiplexing process. Using it one may deduce an optimum receiver structure consisting of a bank of K correlators, with the decision of which message vector to announce being based on the correlator with the greatest output. This model has the decision of which message vector to announce being based on the correlator with the greatest output. This model has the disadvantage in that the number of waveform generators grows very rapidly with the number of channels. Also nothing in general can be said further about the properties of the message waveforms $f_m(t)$. In terms of a vector space analogy, this model produces a great number (K) of vectors (waveforms) to represent the input message vectors. However, it should be possible to simplify the model by projecting these vectors (waveforms) onto a set of orthonormal vectors (waveforms) with the minimum number of dimensions much smaller than K. This derivation results in a much more useful model.

The result of the derivation will be the model shown in Figure 6 . This model will be called the generalized orthomux system since it consists of an ordinary orthomux system preceded by the vector transformation $\vec{M}$. The $0_n(t)$ are a set of J different functions defined on $(0,T)$. If the K different messages are denoted by m_i with $i=1,2,\ldots K$, then $\vec{M}$ is a one-to-one transformation between an input vector $\vec{m}_i$
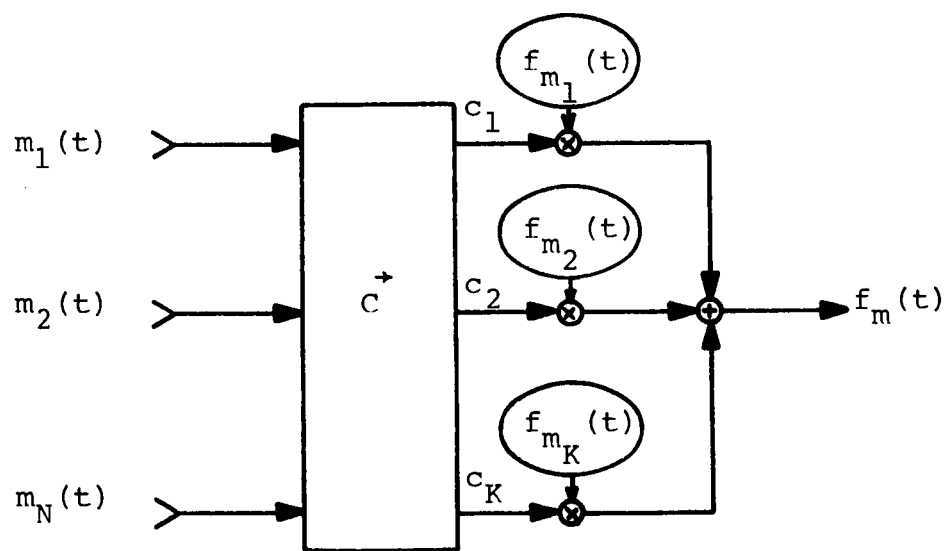
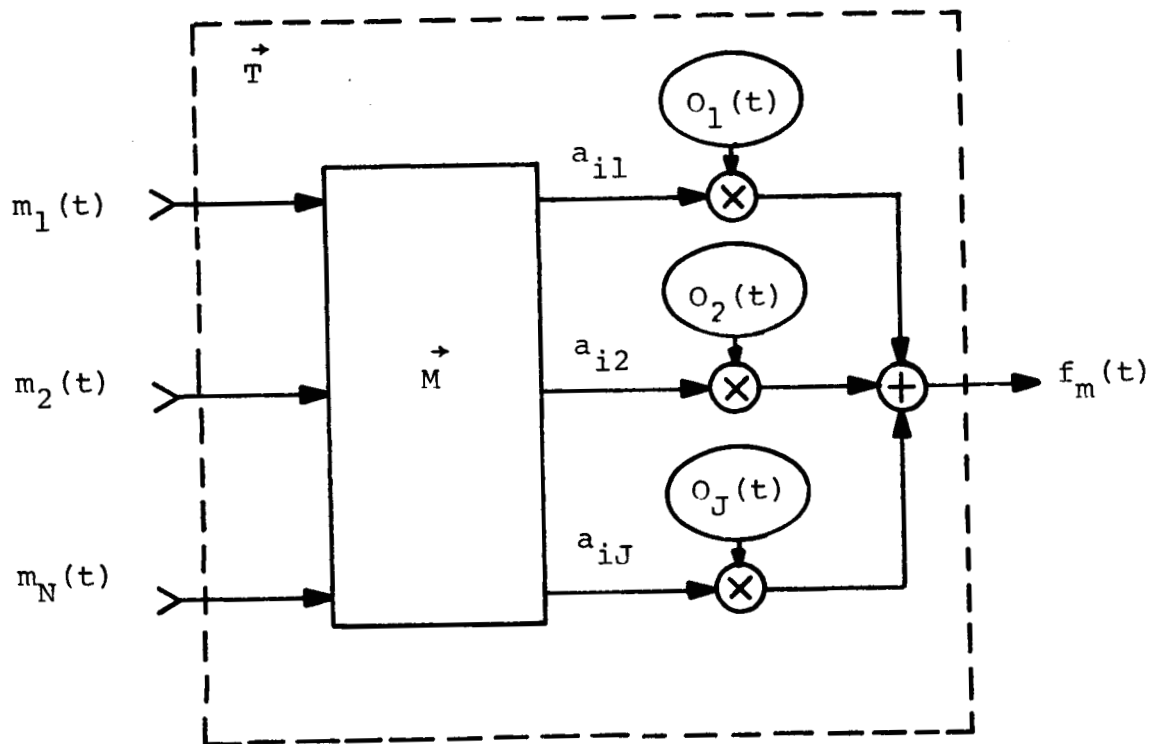Figure 5 .   Primitive model for a general multiplexing system.

Figure 6 . Model for the generalized orthomux system.

and a coefficient vector with J components.

$$\vec{M}(\vec{m}) = \vec{a} \equiv (a_{i1}, a_{i2}, a_{i3}\ldots a_{ij}) \tag{3}$$

$$\vec{M}^{-1}(\vec{a}) = \vec{m} \tag{4}$$

The relationship between the size of J and the size of K will be determined later in this chapter. It is shown there that J will be at most equal to K. The model implies that the output waveform for the ith message can be written:

$$f_{m_i}(t) = \sum_{j=1}^{J} a_{ij} 0_j(t) \text{ for } 0 \leq t \leq T. \tag{5}$$

The implications of this model for multiplexing system analysis and synthesis will be discussed in the next sections. The derivation will be made first. The derivation amounts to a proof by use of the Gram-Schmidt orthogonalization procedure that any set of output waveforms can be expressed by Equation (5).

B. Derivation of Generalized Orthomux Model

Any input message vector may be picked at a starting point. It will be seen that this arbitrariness will cause the representation of Figure 6 to be not unique. This first message vector will be denoted $\vec{m}_1$. The corresponding output waveform is $f_{m_i}(t)$. The first orthonormal function is set equal to $f_{m_1}(t)$, normalized by its energy.

$$0_1(t) \equiv \frac{f_{m_1}(t)}{\sqrt{E_o}}, \text{ for } 0 \leq t \leq T. \tag{6}$$

Where E_o is defined by:

$$E_o \equiv \int_0^T f_{m_i}^2(t) dt = a_{11} \tag{7}$$

The vector $\vec{a}_1$ is $(\sqrt{E_o}, 0, 0, \cdots 0)$.

Next a second message vector is chosen arbitrarily and denoted by $\vec{m}_2$ with its corresponding output waveform $f_{m_2}(t)$. An attempt is made to project $f_{m_2}(t)$ onto the previous orthonormal function:

$$a_{21} \equiv \int_0^T f_{m_2}(t) \, 0_1(t) \, dt, \tag{8}$$

and an auxiliary function $x_1(t)$ is defined to account for the differences between $f_{m_2}(t)$ and its projection on the $0_1(t)$ function.

$$x_1(t) \equiv f_{m_2}(t) - a_{21}O_1(t). \tag{9}$$

If $x_1(t)$ is not zero for all to in the interval $(0,T)$, then a new orthonormal function $O_2(t)$ must be defined at this point. In order for:

$$f_{m_2}(t) = \sum_{j=1}^{2} a_{2j}O_j(t) = a_{21}O_1(t) + a_{22}O_2(t), \tag{10}$$

it is necessary that:

$$a_{22}O_2(t) = x_1(t). \tag{11}$$

Therefore,

$$O_2(t) = \frac{x_1(t)}{\sqrt{E_1}}, \tag{12}$$

where E_1 is the energy of $x_1(t)$:

$$E_1 \int_0^T x_1^2(t)dt = \int_0^T \left[f_{m_2}(t) - a_{21}O_1(t) \right]^2 dt \tag{13}$$

once $b_2(t)$ has been determined, a_{22} is simply the projection of $f_{m_2}(t)$ onto this function:

$$a_{22} = \int_0^T f_{m_2}(t) O_2(t) dt \tag{14}$$

which completes the determination of the quantities required for a two-way message system, $\vec{a}_1$, $\vec{a}_2$, $O_1(t)$ and $O_2(t)$.

In general the same porcedure is used for all the message vectors up to the Kth message vector. If $J-1$ orthonormal functions were necessary to express the previous, $(K-1)^{th}$, output waveform, then:

$$a_{kj} = \int_0^T f_{m_j}(t)O_j(t)dt \text{ for } j=1,2,\ldots K-1 \tag{15}$$

determines all but one of the required coefficients. The auxiliary function is defined:

$$x_k(t) \equiv f_{m_k}(t) - \sum_{j=1}^{J-1} a_{kj\ j}(t). \tag{16}$$

If $X_k(t)$ is not equal to zero at all times in the interval $(0,T)$, it is necessary to introduce another orthonormal function $O_j(t)$:

$$a_{jj}O_j(t) = X_k(t), \tag{17}$$

$$O_j(t) = \frac{X_k(t)}{\sqrt{E_k}} \tag{18}$$

where

$$E_k \equiv \int_0^T X_k^2(t)dt, \tag{19}$$

and
$$a_{jj} = \int_0^T f_{m_k}(t)O_j(t)dt. \tag{20}$$

Therefore all the message waveforms up to the last can be expressed by Equation (5) and the model of Figure 6 will hold for any of the hypothesized multiplexing systems.

C. Required Number of Orthonormal Signal Sources in Model

In the derivation of the proceding section it was shown that all of the output waveforms can be written as an orthonormal expansion:

$$f_{m_i}(t) = \sum_{j=1}^{J} a_{ij} O_j(t) \quad \text{for } i=1,2,\ldots K. \tag{21}$$

In this section the required number J of orthonormal signal sources will be determined.

At each step of the orthogonalization process either one new orthonormal function is required or none. Thus the maximum value of J is K. It will be assumed that at the ith step, $f_{m_i}(t)$ is linearly dependent on the previous $i-1$ functions $f_m(t)$. By the definition of linear dependence this implies that there exists a set of constants b_p such that:

$$f_{m_i}(t) = \sum_{p=1}^{i=1} b_p f_{m_p}(t). \tag{22}$$

The previous functions have been expanded in terms of the orthonormal functions:

$$f_{m_p}(t) = \sum_{j=1}^{J'} a_{p_j} O_j(t) \tag{23}$$

where J' is simply the number of orthonormal functions required for expansion of $f_{m_{i-1}}(t)$. Therefore, by substitution of Equation (23) into Equation (22) :

$$f_{m_i}(t) = \sum_{p=1}^{i=1} b_p \sum_{j=1}^{J'} a_{pj} O_j(t) = \sum_{p=1}^{i=1} \sum_{j=1}^{J'} b_p a_{pj} O_j(t). \tag{24}$$

Then the auxiliary function for this trial is:

$$X_i(t) = \sum_{p=1}^{i=1} \sum_{j=1}^{J'} b_p a_{pj} O_j(t) - \sum_{j=1}^{J'} a_{pj} O_j(t) \tag{25}$$

$$= \sum_{j=1}^{J'} O_j(t) \left[\sum_{p=1}^{i=1} b_p a_{pj} - a_{ij} \right]. \tag{26}$$

the term in the brackets is zero because:

$$\sum_{p=1}^{i=1} b_p a_{pj} = \sum_{p=1}^{i-1} b_p \int_0^T f_{m_p}(t) O_j(t) dt \qquad (27)$$

$$= \int_0^T O_j(t) \sum_{p=1}^{i-1} b_p f_{m_p}(t) = \int_0^T O_j(t) f_{m_i}(t) = a_{ij} \qquad (28)$$

Therefore the auxiliary function X_i is identically equal to zero and no new orthonormal function need be introduced for expansion of $f_{m_i}(t)$.

By induction it is clear that the number of orthonormal functions required for expansion of the K different wave-forms $f_m(t)$ is equal to the number of these waveforms that are linearly independent.

D. Significance of the Model for Analysis of
Multiplexing Systems

Since all multiplexing systems of a very wide class
can be described by the generalized orthomux model, it may
be used for analysis of an unknown multiplexing system.
This might permit reception of otherwise secure transmissions.
The application of the model to the general analysis of
multiplexing system optimality is more important. This topic
will be the subject of the next chapter. However, it will
first be shown below that both of the reported multiplex
systems which do not fit into the orthomux model do fit into
the generalized orthomux model.

1. Amplitude Division Multiplexing

The generalized orthomux model for the amplitude
division multiplexing system descried in Chapter II can be
easily derived. An example system is defined in Figure 7.
One of the four output levels is transmitted during the
interval (0,T). The system has a finite number(two) of
message channels, and each of the messages changes (if they
change)at multiples of T seconds from the time origin.
Thus all the requirements are met for development of a
generalized orthomux model for the system.

Application of the Gram-Schmidt orthogonalization
procedure to the set of four different voltage levels used
for $f_m(t)$ reveals that only one orthonormal waveform is
necessary to represent all the possible $f_m(t)$ waveforms,
and that the coefficients required are those listed in
Figure 7 . This is a somewhat degenerate case, of course.
However, it is certainly the simplest example possible for
demonstration of how the generalized orthomux model works.

2. A Boolean Function Multiplexed System

Titsworth (1963) describes a Boolean function
multiplexing system that has a number of attractive features.
From the block diagram of the system used for implementation
of this system it is difficult to see how it could be
described by the generalized orthomux model. However, the
system has a finite number of channels, each with a finite
number of messages and each of the input messages changes
(if they change) at multiples of T seconds from the time
origin. Thus all of the requirements are met for develop-
ment of a generalized orthomux model for the system.

A simple example is chosen to demonstrate how such
a model can be developed. The system to be analyzed is

a)

$\overline{T}$

m_1	m_2	$f_m(t)$
0	0	V_1
0	1	V_2
1	0	V_3
1	1	V_4

b)

c)

d)

m_1	m_2	O_1
0	0	$V_1 T$
0	1	$V_2 T$
1	0	$V_3 T$
1	1	$V_4 T$

e)
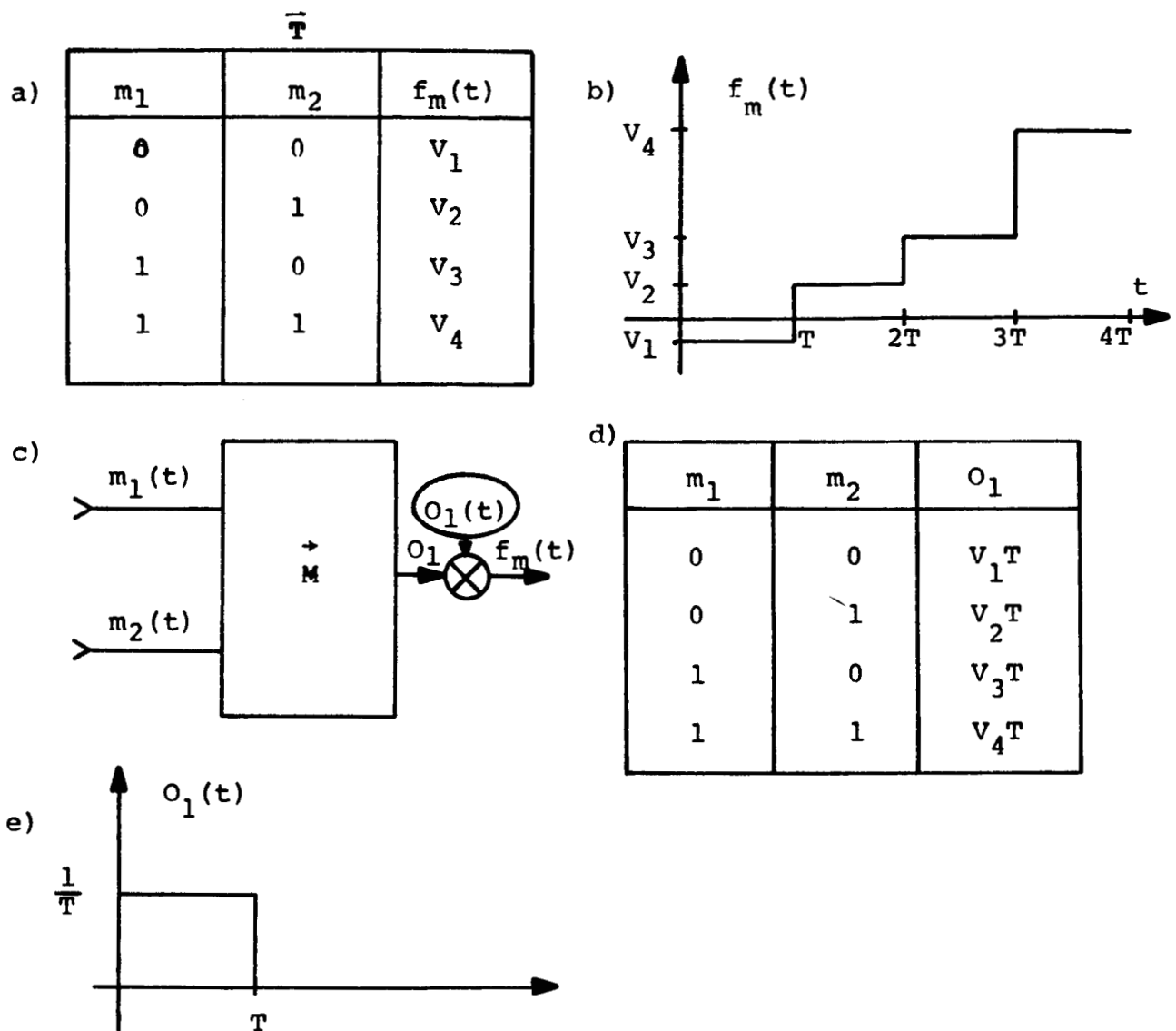
Figure 7 . An example of an amplitude division multiplexing system, a) Transformation table for T. b) Waveform $f_m(t)$ that results from a sequence of inputs: $m_1=m_2=0$; $m_1=0$, $m_2=1$; $m_1=1, m_2=0$; $m_1=m_2=1$. c) Generalized orthomux model. d) Transformation table, $\vec{M}$, for coefficient. e) Orthonormal function.

a)

b) "zero" level $+1$
"one" level -1
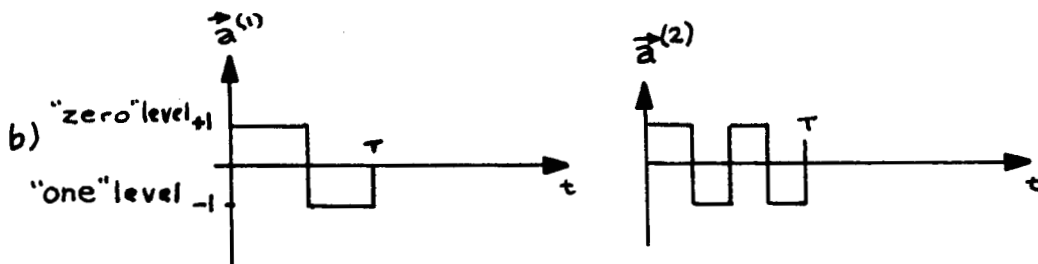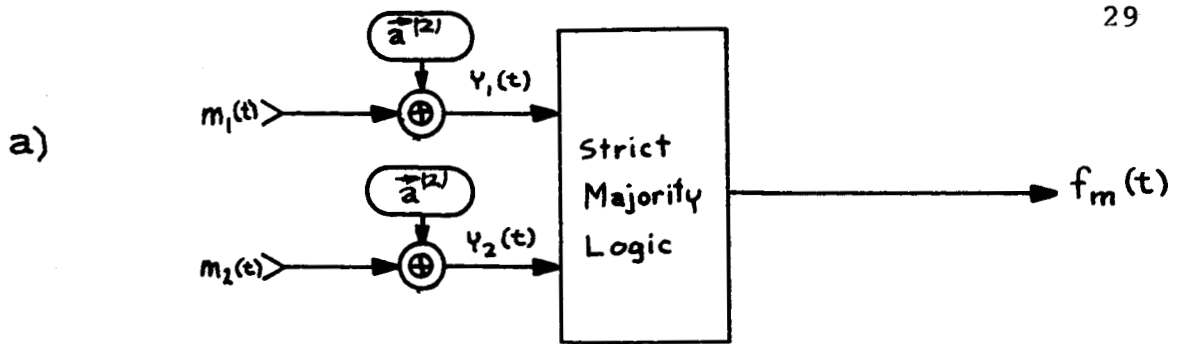
c)

$m_1(t)$	$m_2(t)$	$\Psi_1(t)$	$\Psi_2(t)$	$f_m(t)$
0	0	0011	0101	0111
0	1	0011	1010	1011
1	0	1100	0101	1101
1	1	1100	1010	1110

d)
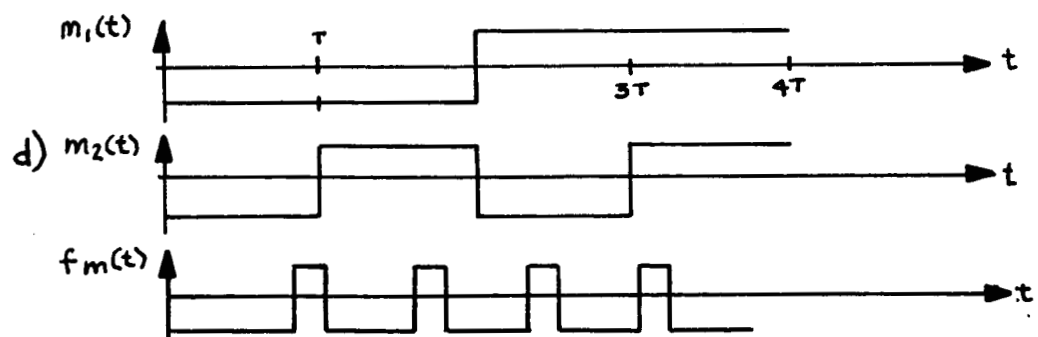
Figure 8. An example of a Boolean-function multiplexed System. a) Block diagram. The exclusive-or function is used for combination of $a^{(1)}$ and m_1. b) Orthogonal functions used in system. c) Transformation table d) Output waveform resulting from an input sequence.

shown in Figure 8. There are binary inputs. The first input is added modulo-2 (the exclusive-or function) to a sequence of binary digits called a [1], using Titsworths's notation. The second input likewise controls a second sequence called a [2]. Titsworth shows that if a [1] and a [2] are orthogonal, zero crosstalk appears at the output of the demultiplexer. Thus in the figure, orthogonal waveforms are specified for a [1] and a [2]. Note that the exclusive-or combination simply means that the sequence a is sent if m=0 and a-complement is sent if m=1. Titsworth also shows that the optimum logic function is strict majority logic. That is, the output $f_m(t)$ is one if there are more ones than zeros at the inputs, and zero if there are fewer.

Titsworth's requirement that a [i] and a [j] be "orthogonal" means that the exclusive-or combination

$$a^{(i)} \oplus a^{(j)}$$

should have an equal number of logical ones and zeros in the sequence representing the code word, where the logic levels are ±1 volts. Equal ones and zeros then means that the integrator output is 0 volts for unlike sequences. Thus the "orthogonality" required by the Boolean function system is indeed the same as conventional definitions of orthogonality.

The generalized orthomux model for this system will now be developed by use of Gram-Schmidt procedure. The notation (-1,1,1,1) will represent a waveform that is -1 volts over the interval from t=0 to t=4 and is +1 volts from t=T/4 to t=T, and so forth. Thus

$$\vec{T}\{\vec{m}_1\} = \vec{T}\{(0,0)\} = f_{m_1}(t) = (-1,1,1,1),$$
$$\vec{T}\{\vec{m}_2\} = \vec{T}\{(0,1)\} = f_{m_2}(t) = (1,-1,1,1),$$
$$\vec{T}\{\vec{m}_3\} = \vec{T}\{(1,0)\} = f_{m_3}(t) = (1,1,-1,1),$$
$$\vec{T}\{\vec{m}_4\} = \vec{T}\{(1,1)\} = f_{m_4}(t) = (1,1,1,-1).$$

The first orthonormal function is defined as:

$$O_1(t) = \frac{f_{m_1}(t)}{\sqrt{E_o}} = \frac{(-1,1,1,1)}{\sqrt{T}} \quad (\frac{-1}{\sqrt{T}}, \frac{+1}{\sqrt{T}}, \frac{+1}{\sqrt{T}}, \frac{+1}{\sqrt{T}})$$

The second waveform is orthogonal to the first, and a new orthonormal basis waveform is required for its representation. Indeed, it is obvious that this is true for all the waveforms so that:

$$O_1(t) = (\frac{-1}{\sqrt{T}}, \frac{+1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}),$$

$$O_2(t) = (\frac{1}{\sqrt{T}}, \frac{-1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}})$$

$$O_3(t) = (\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{-1}{\sqrt{T}}, \frac{1}{\sqrt{T}})$$

$$O_4(t) = (\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{-1}{\sqrt{T}})$$

The transformation $\vec{M}$ is defined by the table below:

$\vec{m}_i$	m_1	m_2	a_{i1}	a_{i2}	a_{i3}	a_{i4}
$\vec{m}_1$	0	0	1	0	0	0
$\vec{m}_2$	0	1	0	1	0	0
$\vec{m}_3$	1	0	0	0	1	0
$\vec{m}_4$	1	1	0	0	0	1

Again it is found that this multiplexing system is a special case of the gneralized orthomux system. Here each of the $f_{m_i}(t)$ waveforms is orthogonal to the others.

E. Significance of the Model for Synthesis of Multiplexing Systems

By use of the generalized orthomux model an infinity
of new multipexing systems can be derived. Not only are
there infinite varieties of orthonormal waveforms for bases,
but an infinity of $\vec{M}$ transformations are available as well.
A short example is presented in this section to demonstrate
the ease by which new multiplexing systems may be designed
by use of the model.

The example system is shown in Figure 9 . The
transformation $\vec{M}$ is defined by the table in part b) of
the figure or the equations below:

$$a_{i1} = m_1 \oplus m_2,$$

$$a_{i2} = m_2,$$

where the special symbol in the first equation represents
the exclusive-or operation. The table reveals that the
transformation is one-to-one. In fact, the inverse trans-
formation $\vec{M}^{-1}$ is given by:

$$m_1 = a_{i1} \oplus a_{i2},$$

$$m_2 = a_{i2}.$$

This transformation is required in the receiver.

Any pair of orthonormal signals can be used to
complete the system, of course. A specific pair is shown
in Figure 9 for an example. A sequence of all possible
input message vectors in succession produces the output wave
shown in part d) of the figure. This waveform demonstrates
that the fundamental requirement for a multiplexing system
is met. That is, each different message vector produces a
different output waveform.

a)

$m_1(t)$

$m_2(t)$

$\vec{M}$

a_{i1}

$O_1(t)$

a_{i2}

$O_2(t)$

$f_m(t)$

$\vec{M}$ Transformation

b)

$\vec{m}_i$	m_1	m_2	a_{i1}	a_{i2}
m_1	0	0	0	0
m_2	0	1	1	1
m_3	1	0	1	0
m_4	1	1	0	1

c)

$O_1(t)$

$\sqrt{\dfrac{2}{T}}$

$T/2$

t

$O_2(t)$

$\sqrt{\dfrac{2}{T}}$

T

t
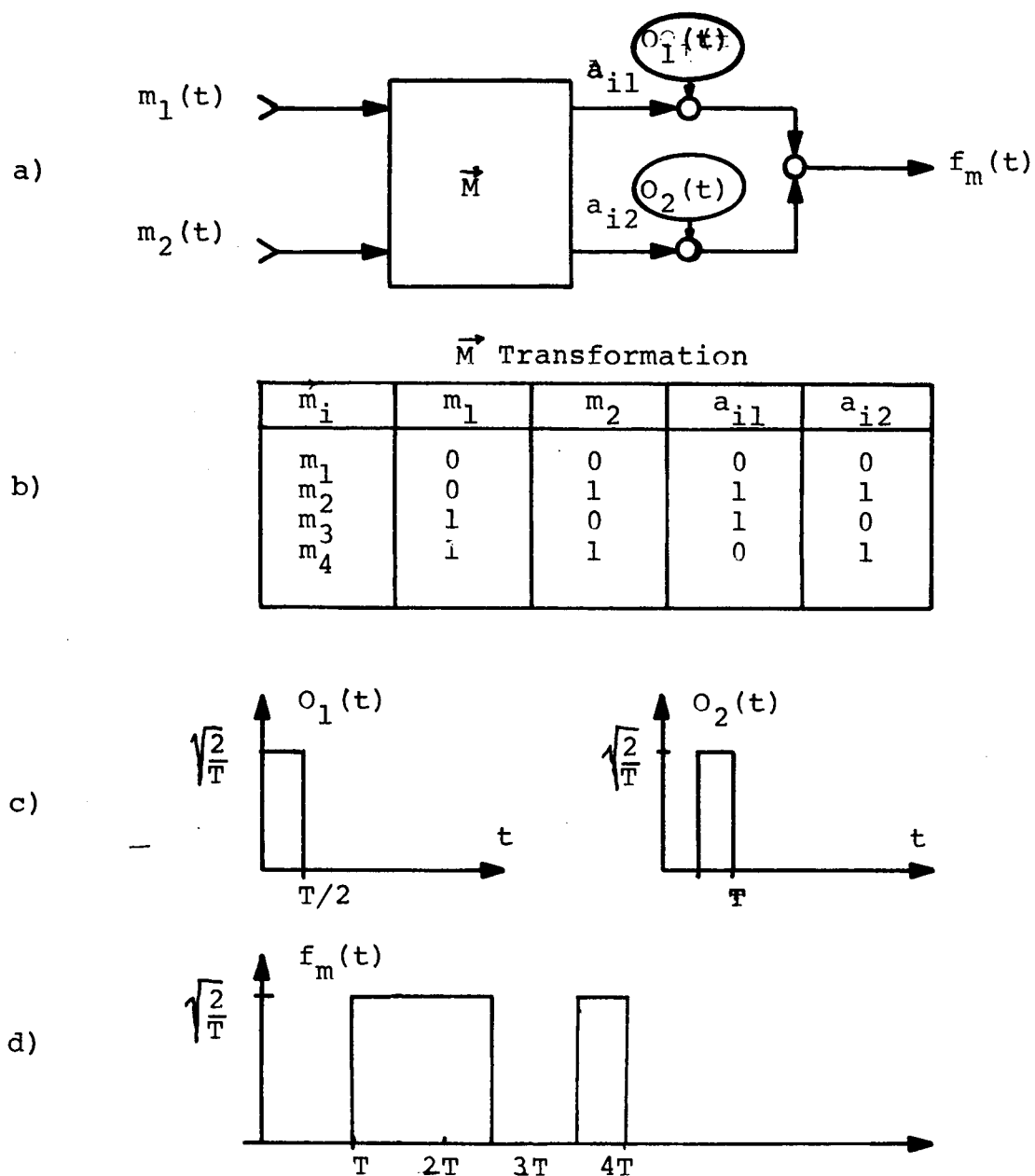
d)

$f_m(t)$

$\sqrt{\dfrac{2}{T}}$

T   2T   3T   4T

Figure 9 . Example of synthesis of a system using the generalized orthomux model. a) Block diagram b) $\vec{M}$ transformation. c) An example of two orthonormal signals. d) The output $f_m(t)$ that results from an input sequence: $\vec{m}_1$, $\vec{m}_2$, $\vec{m}_3$, $\vec{m}_4$.

CHAPTER IV

OPTIMUM MULTIPLEXING SYSTEMS

In this chapter an attempt is made to determine optimum multiplexing systems for various types of channels. The generalized orthomux model developed in Chapter III is used as a mathematical representation of the general multiplexing system. The cases of independent additive white Gaussian noise channel, a bandlimiting channel, a peak signal limiting channel, and a channel with a combination of constraints are considered. The block diagram of such a system is shown in Figure 10.

The Additive Noise Channel

This channel model covers a fairly wide range of actual communications systems. No carrier modulation and demodulation processes are included so that the model can be used directly only for baseband transmission or for systems where the effect of carrier modulation is included by suitable definition of the orthonormal waveforms $O_n(t)$. However, as long as the carrier modulation and demodulation processes introduce negligible distortion in the transmission of $f_m(t)$, this model yields useful results except that it does not allow a study of the effects of frequency and amplitude-limiting in the carrier modulation process. These are discussed later in this chapter.

The additive noise is usually statistically independent of the signal, and the statistics of the noise are frequently assumed to be Gaussian with zero mean. It is further assumed that the noise is white, that is, its spectral density, K, is constant over all frequencies of interest. This, too, is generally valid except when a carrier demodulation process (such as frequency modulation) introduces some complicated spectral density. The probability density function is determined for the output of an orthomux system, because it effectively summarizes the performance of the system for the additive noise channel.

The transmitted signal is:

$$f_m(t) = \sum_{i=1}^{N} m_i(t)O_i(t). \tag{29}$$

Although the calculations are made for an orthomux system, it is clear that simply replacing the $m_i(t)$ weighting factors by a_{ij} would yield corresponding results for a generalized orthomux system. The received signal is:

$$\overset{*}{f}_m(t) = f_m(t) + n(t). \tag{30}$$

The output of the Jth channel is

$$\overset{*}{m}_j(t) = \int_0^T \overset{*}{f}_m(t) O_j(t) dt \qquad (31)$$

Substitution and expansion yields:

$$\overset{*}{m}_j(t) = m_j(t) + \int_0^T n(t) O_j(t) dt \qquad (32)$$

The output probability density function can be obtained from this equation. First, however it is instructive to calculate the mean and variance of the output. Since the mean of the noise is zero, it is apparent that the mean of the last term is zero so that:

$$E[\overset{*}{m}_j(t)] = E[m_j(t)] + E[\int_0^T n(t) O_j(t) dt] = E[m_j(t)] \qquad (33)$$

The mean square value of the output is given by:

$$E[\overset{*}{m}_j^2(t)] = E[\{m_j(t) + \int_0^T n(t) O_j(t) dt\}^2] \qquad (34)$$

The result is:

$$E[\overset{*}{m}_j^2(t)] = E[m_j^2(t)] + \int_0^T \int_0^T R_n(x-t) O_j(t) O_j(x) dt dx \qquad (35)$$

where $R_n(x)$ is the autocorrelation function of the noise. For white noise of density K:

$$R_n(x-t) = K \delta(x-t). \qquad (36)$$

Evaluation of the integral in Equation (35) yields:

$$E[\overset{*}{m}_j^2(t)] = E[m_j^2(t)] + K \qquad (37)$$

The probability density function of the output can be determined from the equation:

$$\overset{*}{m}_j(t) = m_j(t) + \int_0^T n(t) O_j(t) dt \qquad (38)$$

Since any linear transformation of a Gaussian random process yields Gaussian statistics, the probability density function of the output must be Gaussian with the mean and mean square values previously calculated. If $m_j(t)$ is assumed to be a constant, C_j, the result is:

$$p(x) = \frac{1}{\sqrt{2K\pi}}\exp - \frac{(x-C_j)^2}{2K} \qquad (39)$$

The performance is completely independent of the type of orthonormal waveforms used, and thus all such orthomux systems perform equally well for this channel. As mentioned earlier these results are strictly valid for an orthomux system. However essentially the same results apply to the generalized orthomux system if the role of $m_i(t)$ is replaced by the coefficient a_{ij}. The receiver estimates $\hat{a}_{ij}$ of these coefficients also have the Gaussian probability density function with the same mean and variances as previously calculated. Thus the calculated density function effectively summarizes the performance of a generalized orthomux system as well. However, a generalized orthomux system differs in that a single error in reception of a coefficient will in general cause an error in more than one message output.

The above discussion shows that the orthomux system has the desirable property of minimizing interaction or crosstalk between channels. That is, if an error is made in the reception of single a_{ij}, the orthomux system transforms this error into an error in only one of the output message channel, whereas the generalized orthomux system would in general have an error in more than one of the output messages because of the characteristics of the $\hat{M}$ and $\hat{M}^{-1}$ transformations. Since the noise variance is the same for each of the coefficients, a_{ij}, the dependence of output message on a single a_{ij} is desirable. However, the probability of error is also a function of the mean value of a_{ij}, and it is not altogether clear that a properly chosen transformation $\hat{M}$ might lead to an improved performance of all the channels in a system. That is, a trade-off might be possible between crosstalk and the average rate of errors for all the channels.

The theoretical background for this study is available in linear algebra theory (Wozencraft and Jacobs, 1965). It appears that the optimum $\hat{M}$ transformation for combating additive noise is the one that maximizes the vector distance between the different transmitted message waveforms. Thus, coefficients which lead to simplex codes should be chosen, although orthogonal output message waveforms $f_m(t)$ would have essentially the same performance if the number of different messages is large.

For a given $\hat{M}$ transformation the conclusion that all generalized orthomux systems have the same noise performance implies that the only important factor in selection of the orthonormal waveforms is ease of implementation. This point is considered in detail by Williams (1967),but it is worth mentioning here the approaches used.

There are two basic approaches to achievement of equipment simplicity. The first is to select orthonormal waveforms that occur naturally as the impulse response of simple linear, time-invariant circuits. Thus, the convolution operation can be used to avoid multipliers in the transmitter. The same approach can sometimes be used to avoid multipliers in the receiver.

The second approach is to use binary waveforms and digital circuitry. With suitable logic level definitions gates can replace multipliers in both transmitter and receiver.

The Bandlimiting Channel

All practical channels are limited in the bandwidth they can transmit. Signals are also characterized by the bandwidth they occupy in the frequency domain. Signal bandwidth can occasionally be reduced by the removal of redundant information at the source. This problem of "source coding" or "data compression" is a very active area of research in communications at present. However, here the message channels will be characterized by a fixed bandwidth B, which is assumed to be the same for each of the N message channels.

Two approaches are possible to the problem of determining the effects of channel bandwidth constraints on the design of multiplexing systems. First an arbitrary set of orthonormal basis waveforms may be selected, and the crosstalk determined for a channel which limits the bandwidth of the transmitted waveform. The second is to search for waveforms which have minimum bandwidth. The second approach leads to more specific answers and is the one taken here. Williams (1967) investigates the effect of various kinds of band-limiting on some of the orthonormal waveforms that are interesting from an implementation viewpoint.

1. Bandwidth Required for a Noiseless Channel

It is reasonable to suppose that the minimum bandwidth required to transmit N independent message channels, each bandlimited to B Hertz, is NB. However this is not the case if absolutely noise-free channel performance is assumed. In fact it will be shown below that a channel bandwidth of B Hertz will suffice, regardless of the size of N, and that if coding of the individual channels is permitted, the transmission bandwidth may be made arbitrarily small. As N increases, however, the waveforms representing different messages have increasingly large peak power to average power ratios and become more alike so that increasing resolution is required in the receiver. Noise sets an ultimate limit to this resolution and must be considered in some way if meaningful results are to be obtained. Thus, it is more useful to consider simultaneously the constraints of bandlimitation and noise. By considering the noisefree band-limiting channel first, insight is gained into the interaction of the two channel limitations.

If each message channel is ideally bandlimited to B Hertz, the Nyquist sampling theorem states that samples taken at a rate of 2B samples per second will permit exact reconstruction of the original message waveforms. Therefore the sequence of 2B samples per second contains exactly the same information as the original waveform. Now it will be assumed that there are a finite number of possible messages for each message channel. Even if the message waveforms have a continuous range of possible levels, this assumption can be justified by observation of the fact that any practical communication system need transmit message levels to within some tolerance limit.

Since each message has only a finite number of significant levels, the samples from each message channel can be coded into a sequence of binary numbers, with only a finite number of binary digits being required. The binary numbers from all the channels can be combined (by placing them one after another for instance). This binary number can be used as the weight for a waveform used to transmit the entire set of messages. For example if the number is 100,000,000 for a three channel system with each channel having eight or fewer messages, the peak voltage of the waveform could be set at 2^8 microvolts.

The basic waveform can be chosen to occupy a minimum bandwidth. A time function (sine $2\pi Bt)/t$ may be repeated every 1/2B seconds without interference between pulses if the detection is based on sampling the waveform a multiples of 1/2B from the time origin. This is because all but one of the train of pulses is zero at the sampling time. Not only is this set of pulses desirable because of the possibility of zero crosstalk for point detection, but it is optimum in the sense of minimum bandwidth since that its spectrum is flat and entirely contained in a bandwidth of B Hertz.

Thus a construction has been made for a multiplexing system which will transmit N message channels each of bandwidth B in a total bandwidth of B Hertz. The system is entirely feasible as long as the channel is indeed noiseless. Equipment complexity is reasonable if the number of message channels and their possible levels are small.

In the derivation above it has been assumed that the individual channels were sampled at the Nyquist rate and that the system merely coded the set of samples from all the message channels into a single waveform. It is possible to reduce the transmission bandwidth still further by coding the outputs of the message channels. For instance if n of the samples of each message channel are transformed into a single binary number then the transmission bandwidth can be reduced by the same coding scheme to B/n. Thus the transmission bandwidth can be made arbitrarily small by this coding procedure, at the expense of increased peak to average power ratio and/or increased signal resolution difficulties at the receiver.

The coding schemes discussed above are feasible only for a very small number of channels each with a very small number of possible messages. However the results are interesting because it is not

obvious that coding may be used to reduce transmission bandwidth to an arbitrarily small value. Most coding procedures are used for other purposes and have a side effect of increasing bandwidth.

The answer that NB Hertz is the minimum bandwidth for transmission of N channels of bandwidth B is so intuitively appealing that it is desirable to investigate the channel assumptions necessary to arrive at this answer. One set of reasonable assumptions which will lead to this answer is the following:

a) Suppose the system is an orthomux system so that interaction between channels is minimized, as was detailed in the previous section. Therefore,

$$f_m(t) = \sum_{n=1}^{N} m_n(t) O_n(t)$$

b) Since each of the message channels is ideally bandlimited to Hertz it is necessary that $T = 1/2B$.

c) There is a theorem by Landau and Pollack (1962) which allows a bound on the number of orthonormal waveforms $O_n(t)$ possible in an interval of T seconds. The criterion of bandlimiting is somewhat unusual and will lead to a slightly different bound than the NB desired, but it will be clear that the transmission bandwidth must be essentially $W = NB$. The theorem states that the number of orthogonal waveforms that are zero outside of an interval of T seconds is (conservatively) over bounded by:

$$N \leq (2.4)TW$$

where W is the bandwidth defined such that none of the orthogonal waveforms has more than 1/12 of its energy outside the interval:

$$-W \leq f \leq \neq W.$$

From these assumptions and the theorem above the transmission bandwidth W is:

$$W \geq (0.835)NB.$$

and a slightly different criterion of bandlimiting could easily lead to a bound of exactly NB.

For other reasonable criteria of bandlimiting slightly different results will be obtained, but it is clear that W will be always bounded by kNB where k is approximately one. It is easy to see that FDM with single sideband modulation of the subcarriers and super FDM both achieve essentially the minimum bandwidth of NB. Thus, there appears to be very little to be gained in bandwidth conservation by the use of other orthogonal waveforms as basis. It is possible that some other orthonormal waveforms will have essentially the same bandwidth and will be superior in some other respect such as peak to average power ratio or ease of implementation. In a search for such candidates it is natural to examine the waveforms which achieve minimum bandwidth according to various criteria of band-limiting.

Slepian and Pollak (1961) show that the prolate spheroidal wave functions achieve a maximization of the fraction of energy remaining in a fixed bandwidth for a function which is strictly limited to T seconds in the time domain. Curves of these functions are presented by Slepian et al. (1961), but in general they are very similar to (sine x)/x functions in appearance. These orthonormal waveforms are rather undesirable from the ease of implementation viewpoint, but their overall performance parameters are compared to conventional FDM systems in the next chapter.

The Peak Limiting Channel

Because most carrier modulation and demodulation processes have a peak modulation signal limitation, it is necessary to investigate which set of orthonormal basis waveforms are optimum for this channel. Before making this study, however, it is desirable to show why carrier modulators have this modulation signal constraint, since this is not widely realized except for amplitude modulation.

For the various amplitude modulation systems (AM,SSB,DSB, vestigial sideband, etc.) the peak power that must be produced by the transmitter power amplifier is determined by the peak input modulation signal. Such power amplifiers are limited in the amount of peak output power they can produce without damage to themselves or related equipment. This is a well known limitation for amplitude modulation systems, but similar limitations exist for frequency and phase modulation as well.

For frequency modulation the radio frequency bandwidth required is as much a function of the peak modulation voltage as it is a function of the modulation frequency. Indeed the simplest estimate for bandwidth for an FM system is:

$$B = 2(\Omega + \omega_m)$$

where ω_m is the highest frequency component of the modulation signal, and Ω is the peak frequency deviation, which is proportional to the peak modulation voltage. Thus the limitation on radio frequency bandwidth implies a limitation on the peak modulation signal.

The phase modulation systems that are currently in use employ synchronous demodulation in the receiver. That is, the product is formed between the phase modulated carrier and a coherent local oscillator:

$$\cos\{\omega_c t + \emptyset(t)\} \; \sin \omega_c t$$

which after suitable filtering yields a signal proportional to sin $\emptyset(t)$. In order for this sine function to be a one-to-one transformation of the input phase, the peak magnitude of the modulation must be limited to less than 90 degrees. Indeed, if the system is to be linear, the modulation must be restricted to much less than 90 degrees so that the approximation:

$$\sin \emptyset(t) \; \tilde{} \; \emptyset(t),$$

may be made.

Thus, each of the common analog modulation systems has a peak modulation signal limitation. On the other hand it is well known that the output signal to noise ratio is proportional to the average power in the modulation signal. Appendix A demonstrates this fact for the general multiplexing system. Therefore, in any signal waveform design problem it is important to pick waveforms that have a minimum peak to root-mean-square value. Equivalently one can minimize the peak power to average power ratio.

In the general multiplexing system model used here the carrier modulation and demodulation processes are included in the channel. Two approaches are possible in investigating the complications caused by this channel constraint. The first is to determine which waveforms have optimum peak-to-average power ratio. Such a set of waveforms will produce the maximum output signal-to-noise ratio for a channel that has a peak modulation signal constraint. The second approach is to determine the effect of peak clipping on waveforms that are attractive from the point of view of other constraints, such as easy implementation or small bandwidth. Conventional multiplexing systems are designed on such a basis. One calculates the probability of a peak level being exceeded and designs the system so that this happens a certain small fraction of the time. In fact amplitude clippers are often employed to ensure that the modulation signal does not exceed the maximum allowed level.

The first approach leads to a tractable problem, and a set of optional waveforms is easy to determine. The following section is concerned with this approach. The second approach is analytically very difficult because of the non-linear transformations required. Computer simulations, on the other hand, give immediate results for such clipping. Some analytical results and more extensive simulation

results conclude this chapter.

1. Optimal waveforms for minimum peak-to-average power

It is clear that the minimum peak-to-average power ratio for any waveform is one. This fact is almost self-evident, but a mathematical demonstration will serve to provide insight. If $f_m(t)$ is a voltage, the peak power is:

$$P_p = \frac{S^2}{R} \tag{40}$$

where the numerator is the maximum value of $f_m^2(t)$ on the interval $(0,T)$ and the denominator is the resistance level at the point where the voltage appears. The average power is:

$$P_a = \frac{1}{RT} \int_0^T f_m^2(t)\, dt \tag{41}$$

By an elementary theorem of calculus:

$$\int_0^T f_m^2(t)\, dt \le S^2 T \tag{42}$$

Therefore

$$P_p/P_a \ge 1. \tag{43}$$

Furthermore the maximum ratio of one holds only when the equal sign holds in Equation (43). This occurs only when $f_m^2(t)$ attains its maximum value of S^2 at every point in the interval $(0,T)$:

$$f_m^2(t) = S^2 \quad t\varepsilon(0,T) \tag{44}$$

Therefore the optimum signal for a peak amplitude constraint is a binary signal with zero average value:

$$f_m(t) = \pm S \quad t\varepsilon(0,T) \tag{45}$$

The sets of output waveforms satisfying Equation (45) may be divided into classes called synchronous and asynchronous. The term asynchronous is used to denote binary waveforms that may have transitions at any point in the interval $(0,T)$. An example is the so-called random telegraph signal. Synchronous binary waveforms have zero crossing that always occur in response to a master clock in the

system. That is, possible transitions are always at the same instants for all $f_{m_i}(t)$ waveforms, and these possible transition times are at integral multiples of some fraction of the interval T:

$$t_o = \frac{T}{n}$$

Only synchronous waveforms will be considered here because the theory of their performance is well developed, and they are so much easier to implement that asychronous waveforms have apparently never been used in multiplexing systems.

If the output waveforms are restricted to the synchronous case, the problem becomes one of coding. Since all binary waveforms of zero average value are optimal for the peak limiting channel, it is necessary now to consider the other performance criteria of noise performance and bandwidth requirements. There are three major types of block codes suitable for use in a generalized orthomux system. They are the simplex, orthogonal, and bi-orthogonal codes. Simplex codes have the best noise performance, but they are only slightly superior to orthogonal codes if the number of possible output waveforms is large. Orthogonal codes are binary waveforms that are all orthogonal to one another in the same sense that any functions are orthogonal.

The distinction between these codes is best explained in terms of a vector space representation of the possible waveforms. Each possible $f_m(t)$ waveform is called a code word. Each code word is divided into n binary bits that are either +S or -S in amplitude. Thus the code words can be plotted in n dimensional space. Simplex codes are optimal in the sense that the distance between all the code words is as large as possible. Orthogonal codes have code words arranged so that they are orthogonal in a vector sense. Bi-orthogonal codes are formed from a given orthogonal code by use of the negative waveforms of all the code words in the orthogonal code. Thus, bi-orthogonal codes have twice as many possible code words for a given number of dimensions, which implies that they require half the bandwidth of orthogonal codes at the expense of twice as much signal power for a given error probability. Implementation of a bi-orthogonal code requires approximately half as much equipment as an orthogonal code.

The performance of these codes is compared to conventional FDM and TDM in the next chapter.

A Channel With a Combination of Constraints

Although it is necessary to be somewhat specific about the type of channel when it has more than one constraint, it has been possible to determine a parameter which permits study of system optimization for the most important channels. It was initially clear that such a parameter must include simultaneously the peak-to-average power ratio characteristics and bandwidth of the waveform $f_m(t)$, since one

of these quantities could always be improved at the expense of the other. It was decided to investigate the most commonly used channels to determine if such a parameter for the waveform $f_m(t)$ could be determined which would maximize the overall system performance in some meaningful sense.

The block labeled channel in the previous analysis most commonly includes some type of carrier modulation process in order to transfer the spectrum of $f_m(t)$ to a higher frequency for transmission over some physical channel, such as a wire line or radio link. Also included is a carrier demodulator which transfers the spectrum back to low frequencies to recover $f_m(t)$ at the receiver. The carrier modulator almost always has a limitation on the peak amplitude of $f_m(t)$, as previously explained. In the physical channel between the carrier modulator and demodulator the most commonly assumed channel imperfection is that of additive white noise. Carrier modulation methods are compared on the basis of the output signal to noise ratio, and formulas are available for these ratios for the five common carrier modulation methods, in Downing (1964) for instance:

$$\left(\frac{S}{N}\right)_{AM} = \frac{A_C^2 \ \overline{f_m^2(t)}}{2 B_{f_m} N_O}$$

$$\left(\frac{S}{N}\right)_{DSB} = \frac{A_C^2 \ \overline{f_m^2(t)}}{2 B_{f_m} N_O}$$

$$\left(\frac{S}{N}\right)_{SSB} = \frac{A_C^2 \ \overline{f_m^2(t)}}{4 B_{f_m} N_O}$$

$$\left(\frac{S}{N}\right)_{PM} = \frac{A_C^2 \ \theta^2 \ \overline{f_m^2(t)}}{2 B_{f_m} N_O} \qquad \text{(narrow band)}$$

$$\left(\frac{S}{N}\right)_{FM} = \frac{3 A_C^2 \ f_D^2 \ \overline{f_m^2(t)}}{2 B_{f_m}^3 N_O}$$

The one-sided noise spectral density is N_O, A_C is the peak unmodulated carrier amplitude, B_{f_m} is the bandwidth of the modulation signal, θ is the peak phase deviation for the narrowband phase modulation, and f_D is the peak frequency deviation for the frequency modulation case.

By use of a little algebra, the signal to noise ratios for amplitude modulation, double sideband suppressed carrier modulation, and narrow band phase modulation can be shown to be inversely proportional to the product of the bandwidth B_{f_m} and the peak to average power ratio P_{f_m} of the modulation signal $f_m(t)$.

$$\frac{2N_O}{A^2 \hat{f}_m^2(t)} \left(\frac{S}{N}\right)_{AM} = \frac{1}{B_{f_m} P_{f_m}} = \frac{1}{\Gamma_{f_m}}$$

$$\frac{2N_O}{A^2 \hat{f}_m^2(t)} \left(\frac{S}{N}\right)_{DSB} = \frac{1}{B_{f_m} P_{f_m}} = \frac{1}{\Gamma_{f_m}}$$

$$\frac{2N_O}{2A_c^2 \hat{f}_m^2(t)} \left(\frac{S}{N}\right)_{PM} = \frac{1}{B_{f_m} P_{f_m}} = \frac{1}{\Gamma_{f_m}}$$

For channels with a peak limitation the peak modulation $\hat{f}_m(t)$ is fixed and thus to maximize the output signal to noise ratio the product of bandwidth and peak to average power ratio should be minimized. This product is denoted by capital gamma. It has apparently not been considered previously in waveform optimization. For wide band frequency modulation the transmission bandwidth is approximately $2_{f_D} \hat{f}_m(t)$, thus for a fixed bandwidth allocation it is reasonable to maximize:

$$\frac{2N_O}{3A_c^2 \, f_D^2 \, f_m^2(t)} = \frac{1}{P_{f_m} B_{f_m}^3} = \frac{1}{\Gamma_{f_m} B_{f_m}^2}$$

so that for wide band FM the bandwidth B_{f_m} must be weighted more heavily in the output signal to noise ratio minimization. For narrow band FM the output signal to noise ratio is proportional to:

$$\frac{1}{\Gamma_{f_m}} \left(\frac{B_{RF}}{2B_{f_m}} - 1\right)^2$$

where B_{RF} is the allocated transmission bandwidth. For wide band PM and single sideband the results depend on higher order properties of the modulation signal and cannot in general be written in terms of only the bandwidth and peak to average power ratio of $f_m(t)$.

It would have been more satisfying if the output signal to noise ratio had turned out to be maximized by optimization of a single parameter of the multiplexing system output waveform. In three of the most widely used carrier modulation methods, AM, DSB, and narrow band PM, this result is obtained. In wideband FM the

parameter to be optimized is very similar: it differs only in the weighting of the bandwidth. In general the conclusion may be made that for many important channels the optimum multiplexing system is the one that has minimum bandwidth times peak to average power ratio product, Γ_{f_m}.

Before seeking the optimum multiplexing system for this criterion, it is desirable to derive some bounds on the minimum Γ_{f_m} that can be expected. It is possible to do so, but the general bounds turn out to be rather loose or conservative. Research continues on improvement of these bounds, so that the best orthomux systems can be shown to be more nearly optimum.

The bandwidth was previously constrained by:

$$B_{f_m} \geq 0.833 N B_m.$$

The peak to average power ratio was previously shown to be constrained by:

$$P_{f_m} \geq 1$$

This constraint is true for any waveform, of course. It might be expected that it can be tightened for orthomux systems, but the most general bound is:

$$B_{f_m} P_{f_m} \geq 1 \cdot (0.833 N B_m) \equiv \Gamma_o$$

For multiplicative orthomux systems:

$$f_m(t) = \sum_{n=1}^{N} m_n(t) O_n(t)$$

$$\hat{f}_m(t) \geq \hat{m}(t) \sum_{n=1}^{N} |\hat{O}_n(t)| \geq \hat{m}(t) |\hat{O}_n(t)| \geq \hat{m}(t) \frac{1}{\sqrt{T}}.$$

Therefore:

$$P_{f_m} = \frac{\hat{f}_m^2(t)}{\overline{f_m^2(t)}} = \frac{P_m}{N},$$

and:

$$\Gamma_{f_m} \geq 0.833 \Gamma_m. \equiv \Gamma_x$$

This last bound is very loose but it shows that for a multiplicative orthomux system the value of gamma must be proportional to p_m. The existance of both the Γ_x and Γ_o bounds leads one to believe that Γ_m must be bounded by some constant times $N\Gamma_m$, because p_m can be increased so that the Γ_x bound is greater and then Γ_{f_m} must increase linearly with p_m, and the corresponding statement can be made about N. However, multiplication of the Γ_x and Γ_o bounds leads only to:

$$\Gamma_{f_m} \geq 0.833 \ B_m\sqrt{Np_m}.$$

In one important case it is possible to show that the bound increases linearly with N and Γ_m. This is the case for which the peak of $f_m(t)$ is determined by the peak of a single orthonormal function as in multiplicative TDM systems:

$$f_{\hat{m}}^2(t) = \hat{m}^2(t) \ \hat{0}_n^2(t) \geq \hat{m}^2(t) \ \frac{N}{T},$$

where the orthonormal waveform has been confined to T/N seconds. Thus:

$$P_{f_m} \geq P_m,$$

and:

$$\Gamma_{f_m} \geq 0.833 \ N\Gamma_m \equiv \Gamma_z.$$

One final bound will now be derived which will be useful for systems for which it is possible for all the orthonormal waveforms to have their peaks at the same time.

$$f_{\hat{m}}(t) = \hat{m}(t) \sum_{n=1}^{N} |0_n(t)| \geq \hat{m}(t) \ N|0_n(t)|_{MIN}$$

CHAPTER V

PERFORMANCE COMPARISONS AND CONCLUSIONS

Performance Comparisons

The performance comparison between the various multiplexing systems are very efficiently made by use of Table 4. The top three lines are the most important bounds. The next group includes common FDM systems. Following the corresponding TDM systems are two of the orthomux systems that are attractive from the viewpoint of equipment simplicity. Actually the last TDM system, PCM TDM, is representative of systems that have optimum peak-to-average power ratio. The first column is an estimate of the required transmission bandwidth. For many systems, the bandwidth can be only estimated according to some natural criterion, since the spectrum of a time limited signal extends over all frequencies. The next column is the peak-to-average power ratio of the composite $f_m(t)$. The worst case peak value is taken even though this may occur rarely. The third column is the product of bandwidth and peak-to-average power ratio, which was shown in Chapter IV to be a meaningful way to compare the performance of systems for many channels. The fourth column is the ratio of the value of gamma for the system to the general bound derived in Chapter IV. An optimum system would have a ratio of one, and the fact that the ratios for all the systems considered are much greater lead one to believe that the general bound is not the tightest one that could be derived. The other bounds in the table are the additive bound Γ_1 for those multiplicative orthomux systems for which it is possible for all the orthonormal waveforms to add at the same time to produce a peak, and the single function bound Γ_2 for those multiplicative orthomux systems for which the peak is determined by the peak of a single orthonormal function. These bounds, when used for comparison with the appropriate systems in the fifth column, yield very informative results.

For a channel which disturbs the $f_m(t)$ only by the addition of independent, white Gaussian noise, the binary orthomux and the real exponential orthomux systems seem to be optimum from the viewpoint of the hardware simplicity criterion. The table demonstrates that the real exponential set produces a rather inferior multiplexing system from the standpoint of bandwidth and peak-to-average power ratio. The binary orthomux system is much more attractive from these standpoints. For the band-limiting channel, the prolate spheroidal waveforms lead to an optimum system. These waveforms achieve the minimum possible bandwidth, as has previously been discussed. However, it is very difficult to evaluate their other performance parameters, such as peak-to-average power ratio because these cannot be expressed in terms of elementary functions and can only be tabulated. They do not appear to be feasible for use in a practical multiplexing system because FDM systems, which are simply to build and analyze, achieve essentially the same bandwidth. The table shows that ideal FDM is a very high performance system from the peak-to-average viewpoint as well. For a channel which limits the peak amplitude of the composite

Table 4   Performance comparisons for important multiplexing systems

MULTIPLEXING METHOD	Bf_m Transmission Bandwidth	Pf_m Peak-to-Average Ratio	f_m Bandwidth Peak Product	Γ_{f_m}/Γ_o Comparison With General Bound	Γ_{f_m}/Γ_i Comparison With Special Bound
General Bound- Γ_o	$0.833NB_m$	1	$0.833NB_m$	1	--
Additive Bound- Γ_1	$0.833NB_m$	NP_m	$0.833N^2\Gamma_m$	NP_m	--
Single Function Bound- Γ_2	$0.833NB_m$	P_m	$0.833N\Gamma_m$	P_m	--
DSB-FDM	$2NB_m$	$2NP_m$	$4N^2\Gamma_m$	$4.7NP_m$	$4.7 \; (\Gamma_1)$
Ideal-FDM	NB_m	$.65NP_m$	$.65N^2\Gamma_m$	$0.78NP_m$	--
AM-FDM	$2NB_m$	$\dfrac{8N}{1+1/P_m}$	$\dfrac{16N^2\Gamma_m}{1+P_m}$	$\dfrac{19.2NP_m}{1+P_m}$	--
FM-FDM (Narrow Band)	$2NB_m$	$2N$	$4N^2B_m$	$4.7N$	--
PAM-TDM (Square Pulses)	$4NB_m$	$2P_m$	$8N\Gamma_m$	$9.6P_m$	$9.6(\Gamma_2)$
PCM TDM (Square Pulses)	$2nNB_m$	1	$2nNB_m = NB_m\log\left(\dfrac{P_mRq}{3}\right)$	$2.5h = 1.2\log\dfrac{P_mRq}{3}$	--
Binary Orthomux	$2NB_m$	NP_m	$2N^2\Gamma_m$	$2.5NP_m$	$2.5 \; (\Gamma_1)$
Real Exponential Orthomux	$>6NB_m$	$>3N^3P_m$	$>18N^4\Gamma_m$	$21.5N^3P_m$	$>21.5N^2 \; (\Gamma_2)$

$f_m(t)$, coded systems such as the PCM TDM system shown in the table are optimum. For a channel with a combination of constraints it is difficult to say in general which system is optimum. In view of the difficulty in calculating the peak value of a sum of arbitrary orthonormal waveforms, the only feasible approach is to derive bounds for the gamma parameter, and then to compare the gammas of various systems to this bound. Then if a system can be found which comes close enough for all practical purposes, the problem is solved. Unfortunately the general bound previously derived is so conservative that it has not been possible to find a system which comes close to it. The two specialized bounds shown in the table are very useful however. They show that the binary orthomux system is about as good as can be obtained in a system in which it is possible for all the orthonormal signals to add in phase to produce a peak. Likewise PAM TDM with square pulses is seen to be a fairly good system when compared to the bound for systems which have a peak determined by a single orthonormal signal.

Conclusions

A general model has been derived to describe all possible multiplexing systems except those for which the messages determine the value of the output in a non-real-time manner. In addition to its use in analysis, the model was shown to be capable of generating many new types of multiplexing systems. One very interesting multiplexing system thus derived is "super FDM" which used both sine and cosine waves of the same frequency as subcarriers. This system can be shown to have a low peak-to-average power ratio and to achieve essentially the minimum possible bandwidth.

The multiplexing system model allows a study of optimality of multiplexing systems for various channels. In case of a channel with additive independent white A Gaussian noise, all orthogonal waveforms lead to equal performance. Thus, the subjective criterion of the ease of implementation becomes the only important standard of comparison. The system based on the real exponential set and the binary orthomux system are probably as simple to implement as can be achieved. For a bandlimiting channel a bound was derived on the minimum bandwidth that can be achieved with a multiplexing system. Prolate spheriodal waveforms achieve this bound, but have a number of other features that make them unattractive for multiplexing applications. Ideal FDM and SSB FDM achieve almost as small a bandwidth and are superior to prolate spheroidal waveforms in several other respects. Binary waveforms were shown to be optimum for a peak limiting channel. Once these waveforms have been selected for this channel, it remains only to select the type of code to be used. Simplex codes are optimum but orthogonal codes are essentially as good if the number of possible waveforms is large.

It is difficult to determine which multiplexing system is optimum for a channel with a combination of constraints. A different approach was taken here to yield a realistic standard of comparison for many practical communications channels. For a channel

in which double sideband, amplitude, or narrowband phase modulation is used for carrier modulation, it was shown that the overall output signal-to-noise ratio was maximized if a parameter Γ_{f_m} was minimized. This parameter is the product of the peak-to-average power and the bandwidth of $f_m(t)$. Various bounds were derived for this parameter which permit comparison of a given multiplexing system's performance to ideal performance.

Research is now underway to apply these results to the design of a deep-space communication system.

BIBLIOGRAPHY

Anderson, D.R. et al. "Control of Overmodulation with Multiplexed Frequencies," IRE National Convention Record, Vol. 8, (March, 1961), 81-89.

Baghdady, E.J. "New Developments in FM Reception and Their Application to a System of Power Division Multiplexing," IRE Transactions on Communications Systems, Vol. CS-7, No.3 (September, 1959), 147-161.

Ballard, A.H. "A New Concept for Multiplexing Communications Signals," Conference Proceedings--National Convention on Military Electronics, Vol. 6 (June, 1962), 355-362.

_____. "A New Multiplexing Technique for Telemetry," Proceedings of the National Telemetry Conference, Vol. 6-2 (May, 1962), 6-22; 6-8.

_____. "Orthogonal Multiplexing," Space/Aeronautics Vol. 46, No. 6 (November, 1962), 119-121.

_____. "Telemetry Multiplexing with Orthogonal Pulse Waveforms," Proceedings of the National Telemetry Conference, Vol. 10-2 (May, 1963).

Bennet, W.R. "Time Division Multiplexing System," Bell System Technical Journal, Vol. 27 (April, 1941), 199-221.

Bennet, W.R., H.R. Curtis and S.O. Rice. "Interchannel Interference in FM and PM Systems," Bell System Technical Journal, Vol. 34, No.5 (May, 1955), 601-636.

Black, H.S. Modulation Theory. Princeton, New Jersey: Van Nostrand, 1953.

Brock, R.L. and R.C. McCarty. "On the Modulation Levels in a Frequency-Multiplexed Communication System by Statistical Methods," IRE Transactions on Information Theory, Vol. IT-1 (March, 1955), 63-68.

Downing, J.J. Modulation Systems and Noise. New York: Prentice Hall, 1964.

Filipowski, R.F. and L. Bickford. Space Communications
and Applications, Vol. 01: NASA Special Publication
70022, 1965.

Florman, E.F. and J.J. Tary. Regional Signal-to-Noise
Ratios for Multichannel Radio Communications Systems:
NBS Technical Note No. 100, July, 1960.

Judge, W.S. "Multiplexing Using Quasiorthogonal Binary
Functions," Communications and Electronics, Vol. 81,
No.60 (May, 1962), 81-83.

Karp, S. and P.K. Higuchi. "An Orthogonal Multiplexed
Communications System Using Modified Hermite Poly-
nomials," Proceedings of the International Tele-
metry Conference, Vol. I (September, 1963), 341-353.

King, O.B. "Multiplexing Techniques for Satellite
Applications," Electronics, Vol. 32, No.10
(October, 1959), 58-62.

Korobkov, L.A. (translator). "Method of Multichannel
Communications," Transaction No. MCL-893: Wright
Patterson AFB, AD 258, September, 1955.

Landau, H.L. and H.O. Pollock, "Prolate Spheroidal Wave
Functions, Fourier Analysis and Uncertainty-III:
The Dimension of the Space of Essentially Time-and
Band-Limited Signals," Bell System Technical Journal,
Vol. 41, No.7 (July, 1962), 1295-1336.

Landon, V.D. "Theoretical Analysis of Various Systems of
Multiplex Transmission," RCA Review. Vol. 9, No.6
(June, 1948), 287-351; RCA Review, Vol. 9, No.9
(September, 1948), 433-482.

Marcateli, E.A. "Time and Frequency Crosstalk in Pulse-
Modulated Systems," Bell System Technical Journal,
Vol. 30, No.3 (May, 1961), 451-470.

Marchand, N. "Analysis of Multiplexing and Signal Detection
by Function Theory," IRE Convention Record, 1953.

Marchand, N. and H.R. Holloway. Multiplexing by Orthogonal
Functions. Dayton, Ohio: IRE Conference on Airborne
Electronics, Vol. 6-3. May, 1951.

Medhurst, R.G. "RF Bandwidth of Frequency-Division Multiplex Ssytems Using Frequency Modulation," Proceedings of the IRE, Vol. 44, No. 2 (February, 1956), 189-199.

_____. "RF Spectra and Interfering Carrier Distortion in FM Trunk Radio Systems with Low Modulation Ratios," IRE Transactions on Communications Systems, Vol. CS-9, No. 2 (June, 1961), 107-115.

Moskowitz, S., L. Diven and L. Freit, "Cross-Talk Considerations in Time-Division Multiplex Systems," Proceedings of the IRE, Vol. 38, No. 11 (November, 1950), 1330-1335.

Nichols, M.H. and L. L. Rauch. Radio Telemetry. New York: Wiley, 1956.

Nichols "Comparison of Required Radio Frequency Power in Different Methods of Multiplexing and Modulation," IRE National Convention Record, No. 5,(1954), 59-65.

Parry, C.A. "The Equalization of Baseband Noise in Multichannel FM Radio Systems," Proceedings of the IRE, Vol. 45, No. 11 (November, 1957), 1527-1534.

Rowe, H.A. Signals and Noise in Communication Systems. Princeton, New Jersey: Van Nostrand, 1965.

Sanders, R.W. "Communications Efficiency Comparison of Several Communications Systems," Proceedings of the IRE, Vol. 48, No. 4 (April, 1960), 575-588.

Shelton, R.D. "Estimation of Signal-to-Noise Ratio," Southwestern IEEE Conference, Dallas, 1965, (unpublished).

Shelton, R.D. and T. Williams. "Orthogonal Multiplexing Systems Based on Easily-Generated Waveforms," Southwestern IEEE Conference (Dallas, 1967), 20-2-1 to 20-2-8.

Shelton, R.D., T. Williams and S. Riter. "Analog and Digital Simulations of Multiplex System Performance," IEEE International Communications Conference, Paper No. 23.1, Minneapolis, 1967.

Shelton, R.D. and S.Riter "Communications Systems for
    Manned Interplanetary Explorations," Conference
    Proceedings of the 1967 National Symposium of
    the American Astronautical Society. Also to be
    published in Advances in Astronautical Science,
    Vol.24.

Slepian, D. and H.O. Pollak, "Prolate Spheroidal Wave
    Function,Fourier Analysis, and Uncertainty-I,"
    Bell System Technical Journal, Vol. 40 (1961), 43.

Stilz, H. Aerospace Telemetry. Englewood Cliffs, New
    Jersey: Prentice Hall, 1961.

Teplyakov, I.M. "Transmission of Information by Means of
    Orthogonal Signals," Radio Engineering, Vol. 18,
    No.4 (April, 1963), 29.

Titsworth, R.C. "A Boolean Function Multiplexed Telemetry
    System," IEEE Transaction on Space Electronics
    and Telemetry, Vol. SET-9, No.2 (June, 1963), 78-83.

Watt, A.D. et al. "Performance of Some Radio Systems in
    the Presence Thermal and Atmospheric Noise,"
    Proceedings of the IRE, Vol. 46, NO.12 (December,
    1958), 1914-1923.

White, W.D. "Theoretical Aspects of Asynchronous Multi-
    plexing," Proceedings of the IRE, Vol. 38, No.3
(March, 1950), 270-274.

Williams, T. Realization of Optimum Multiplexing Systems,
    Ph.D. Dissertation, University of Houston, to be
    completed in 1967.

Wozencroft, J.M. and I.M. Jacobs. Principles of Communi-
    cations Engineering. New York: Wiley, 1965.

Zadeh, L.A. and K.S. Miller. "Fundamental Aspects of
    Linear Multiplexing," Proceedings of the IRE,
    Vol. 40, No.9 (September, 1952), 1091-1097.

PART II

A Synopsis of

"AN OPTIMUM MULTIPLEXING SYSTEM FOR SPACE COMMUNICATIONS"

by

Stephen Riter

ABSTRACT

One of the limiting factors in the design of a space mission is the performance of the communications system available to support the mission. This report examines the expected state of the art in spacecraft-to-ground communications for the period from 1970 to 1975 in order to determine the communications system performance for possible manned interplanetary exploration missions. This was accomplished by first studying in detail the hardware capabilities for the period of interest in order to determine the hardware performance that can be expected from equipment at this time.

A system (patterned after the Unified S-band communications system which is currently being used for Apollo) was designed, using the predicted equipment performance. The system was then optimized to yield the maximum possible information transfer rate with positive performance margin. An all-digital system was then designed, using coherent phase shift keying (PSK) and again optimized to yield maximum information transfer. It was then determined that the all-digital PSK system gives a considerable theoretical advantage over the optimum Unified S-band system.

I. Introduction

Upon completion of the Apollo Program numerous manned space programs seem equally attractive; among them manned orbital laboratories, Lunar exploration programs, and manned missions to our nearest planetary neighbors, Mars and Venus. From a communications point of view however, the most interesting and challenging of these are the manned interplanetary missions. Although it is presently impossible to pinpoint the exact time for these missions, it appears that such a mission could take place in the mid 1970's which would require a completed communications system design by approximately 1973. Based upon experience gained from Apollo there would be as a minimum a requirement for continuous ranging, tracking, telemetry and voice communications with the spacecraft to planetary distances, $100(10)^6$ nautical miles. This report is primarily concerned with determining the characteristics of equipment available to support such a mission, the performance of an optimum conventional modulation system to meet the communications requirements using this equipment, and the advantages to be gained by using a theoretically optimum communications system, i.e. an all digital phase shift keyed system.

II. Equipment Capabilities

Studies of predicted equipment capabilities in 1973 were made. The results are summarized in Table 1 where a comparison with the performance of present day equipment is presented. These estimates are considered to be very conservative. In all instances the estimates are based on conventional devices which exist at the present time and merely represent extensions of the present state of the art. It is hoped and expected that several breakthroughs in techniques would make certain unconventional devices available at the time of the mission; however, it is not feasible to consider them in a communications system design.

The frequency chosen was in the vicinity of 2.2 GHz, which is the frequency presently used for the primary spacecraft-to-ground communications links. It has been shown (Grimm, 1959) that the 1 to 3 GHz range is the optimum radio frequency for operation of "external-noise-limited" ground based receiving systems. This is because the sum of the two main sources of noise, cosmic noise and noise due to an absorbing atmosphere, exhibits a minimum in this range. In addition Easterling and Goldstein (1965) have shown that there are no "deleterious effects on communications with or tracking of a spacecraft" at Martian distances, using frequencies in this range. Finally, since a tremendous investment in time and money has been made to develop equipment for the Apollo Program to operate at these frequencies, it seems unlikely (discounting a breakthrough in optical communications) that the present frequencies would be discarded.

The primary limitation on spacecraft transmitter power is dictated by transmitter weight, volume, and input power. S-band amplitrons suitable for spacecraft applications with output powers up to 200 watts have been developed by the Raytheon Corporation. These units have a weight, volume, and input power within that allowable for the manned interplanetary spacecraft (North Aviation Report No. SID 64-1-3, 1964). Extension beyond this output power does not appear feasible at this time, within the reliability constraints present for such a long mission.

There appears to be no limit on the amount of power that can be used on the Earth-to-spacecraft link. The value of 400 kw was chosen, based upon data published in Jet Propulsion Laboratory Report No. 32-501, August 20, 1963.

A parabolic antenna with a gain of 60 db at S-band is presently available, although it will not be used for the Apollo lunar landing program. A spacecraft antenna with a gain of 28.6 db is presently under development. Reports indicate (North American Aviation Report No. SID 65-761-3A, 1965) that the same design could be improved to give a gain of 38 db by 1970.

It can be shown that the effective noise temperature of a receiving system is given by:

$$T_{eq} = T_r + T_{sky}/L + T_{line} \left(1 - \frac{1}{L}\right) \qquad (1)$$

where T_r is the receiver equivalent noise temperature, T_{sky} is the equivalent noise temperature of the antenna and sky, T_{line} is the equivalent noise temperature of the radio frequency line, and L the radio frequency line loss factor. At S-band T_{sky} is approximately equal to 6°K (Grimm, 1959). The 60 db parabolic antenna has a noise temperature of approximately 15°K (Filipowsky and Muehldorf, 1965). Traveling wave masers suitable for use in ground receiving stations have been developed with noise temperatures as low as 3.5°K (Tabor and Sibilia, 1963). For the above values and a radio frequency line loss factor of 1.01, the approximate value in present receiving stations Equation (1) yields $T_{eq} = 27°K$.

Jet Propulsion Laboratory Report No. 32-501, indicates that the use of a parametric amplifier as a spacecraft receiver is entirely feasible. Such a receiver would have a total effective system noise temperature of 150°K.

III. Optimization of a Conventional Multiplexing System

A. Optimization Criteria

The performance of a single communications link is usually evaluated in terms of the range equation, which expressed in

TABLE 1

PREDICTED SYSTEM CHARACTERISTICS

Spacecraft to Earth

	Present	1973
Frequency	2.2 GHz	2.2GHz
Transmitter Power	20 watts	200 watts
Transmitter Antenna Gain	28.6 db	38 db
Receiver Antenna Gain	52 db	60 db
Receiver System Temp.	270°K	27°K

Earth to Spacecraft

	Present	1973
Frequency	2.2GHz	2.2GHz
Transmitter Power	10 kw	400 kw
Transmitter Antenna Gain	52 db	60 db
Receiver Antenna Gain	28.6 db	38 db
Receiver System Temp.	4600°K	150°K

decibel notation is:

$$S/N = P_M + G_M - 10 \log_{10} (KBT)$$

$$-20 \log_{10} f -20 \log_{10} R + 37.8 \qquad (2)$$

where P_M = Transmitted power

G_M = Transmitting antenna gain

G_R = Receiving antenna gain

K = Boltzman's constant

B = System bandwidth

T = Effective system noise temperature

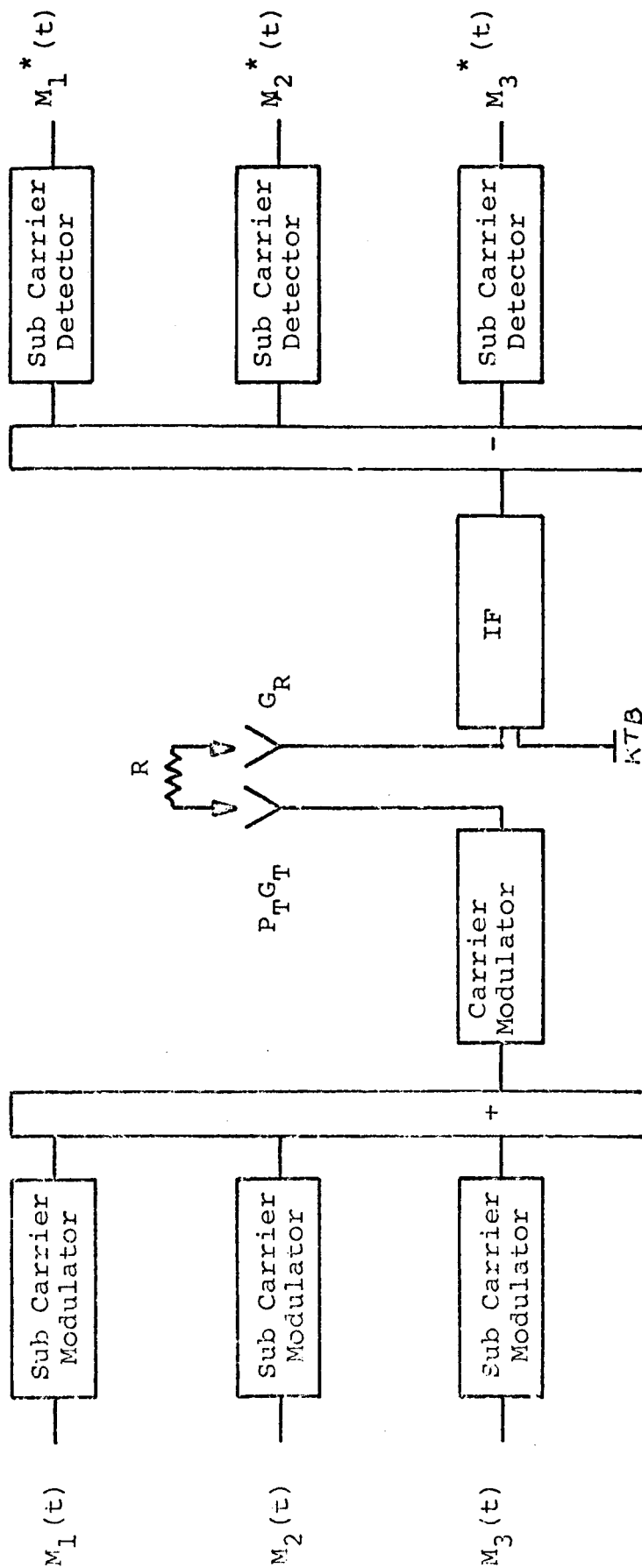R = Range in nautical miles

f = Frequency in MHz

This figure is then compared with S/N required for satisfactory performance to obtain a quantity commonly referred to as performance margin M, where

$$M = (S/N)_{ACTUAL} - (S/N)_{REQUIRED} \qquad (3)$$

Consider the Multiplex link shown in Figure 1. The range equation can be applied to this link by considering the multiplexing process as a scheme which allots certain fractions of the available power to each channel. This division of the available power can be taken into account of in Equation (2) by the addition of 10 $\log_{10}P$, where P is the fraction of power allocated to each channel. The resulting signal-to-noise ratio can then be calculated for each channel of the system and compared with the required signal-to-noise ratio to obtain the margin for each channel.

The equations for each channel can then be grouped together to form the following system of equations which completely characterize the system.

$$A - B_1 + 10 \log_{10} P_1 -(S/N)_{REQ1} = M_1 \qquad (4)$$

$$A - B_2 + 10 \log_{10} P_2 -(S/N)_{REQ2} = M_2$$

$$\cdot \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$A - B_n + 10 \log_{10} P_n -(S/N)_{REQn} = M_n$$

$M_1{}^*(t)$

$M_2{}^*(t)$

$M_3{}^*(t)$

Sub Carrier
Detector

Sub Carrier
Detector

Sub Carrier
Detector

IF

G_R

KTB

R

$P_T G_T$

Carrier
Modulator

Sub Carrier
Modulator

Sub Carrier
Modulator

Sub Carrier
Modulator

$M_1(t)$

$M_2(t)$

$M_3(t)$

R=losses due to range
P_T=transmitter power
G_T=transmitter antenna gain
G_R=receiver antenna gain

Figure 1

A MULTIPLEX LINK

where   A = Sum of losses and gains common to all channels

B_n = Sum of losses and gains common to the nth channel

P_n = Fraction of energy in the nth channel

$(S/N)_{REQn}$ = Required signal-to-noise ratio in the nth channel

M_n = Performance margin of the nth channel.

The common gains and losses would be quantities such as transmitted power, transmitter antenna gain, and receiver antenna gain. The B functions would include quantities such as equivalent noise bandwidths, information rates, and improvement due to modulation.

The optimization criterion chosen for the system of equations should be that the variables B_n and P_n are chosen so that the set of M_n's are all positive and that the minimum M_n is as large as possible for the most extreme conditions.

    B.   Signal Design

An analysis of various conventional modulation and multiplexing techniques indicated that the most attractive approach is to use a unified carrier system where the telemetry and voice channels are modulated on sine wave subcarriers which are then narrow band phase modulated on a carrier along with the ranging information. With this type of system the optimization reduces to finding the optimum division of power as determined by the modulation indices and the maximum telemetry bit rate.

The receiving equipment currently used by NASA are listed in Table 2   with the input signal-to-noise ratio, (S/N), required for satisfactory operation and the equivalent noise bandwidth. The signal performance margin was calculated based on the values of Table 2 for an unmodulated carrier transmitted from the ground to the spacecraft, the up-link, and from the spacecraft to the ground, the down-link, using the range equation. These calculations are tabulated in Table 3 and indicate that the spacecraft-to-ground link was the weaker of the two by a factor of 25 db, primarily because of the available power of 400 kw and 400 watts at the ground and spacecraft respectively. Since the losses of the up-link and down-link information channels are about the same, an optimum waveform for the down-link was selected and then used for the up-link.

A system block diagram of a ground station receiving is shown in Figure 2. A narrow-band phase lock loop is used as a carrier tracking loop to derive a coherent reference signal which is then used to synchronously demodulate the information channels. Bandpass

TABLE 2

PERFORMANCE OF NASA RECEIVING EQUIPMENT

NASA Document No. MH01-13001-414, 1965

Ground Station

Function	$(S/N)_{REQ}$	Equivalent Bandwidth
Carrier Tracking	12 db	700 Hz
PCM TM Detector	9 db	150 KHz 60 KHz 6 KHz
FM Voice Detector	10 db	20 KHz
Ranging Receiver	32 db	1 Hz

Spacecraft

Function	$(S/N)_{REQ}$	Equivalent Bandwidth
Carrier Tracking	12 db	320 Hz
Command Detector	10 db	20 KHz
Voice Detector	10 db	20 KHz

TABLE 3

CALCULATIONS OF SIGNAL-TO-NOISE RATIO OF AN UNMODULATED CARRIER

|	Spacecraft to Ground		Ground to Spacecraft		
	Units	db	Units	db	Source
(1) Range	10^8 nm	-	10^8 nm	-	Assumed
(2) Frequency	2.2 GHz	-	2.2 GHz	-	Table 1
(3) Path Loss	-	-264	-	-264	Eq. E-6
(4) Transmitted Power	200 w	23	400 kw	56	Table 1
(5) Transmitter Antenna Gain	-	38	-	60	Table 1
(6) Receiver Antenna Gain	-	60	-	38	Table 1
(7) Negative System Tolerances	-	-3	-	-6	Assumed
(8) Received Signal Power	-	-140	-	-110	(3)+(4)+(5)+(6)+(7)
(9) System Noise Temperature	$26^\circ K$	-	$150^\circ K$	-	Table 1
(10) Noise Spectral Density	-	-215	-	-207	$10\log_{10}K_0 T$
(11) Bandwidth	700 Hz	28	320 Hz	25	Table 2
(12) Noise Power	-	-187	-	-182	(10)+(11)
(13) (S/N)	-	47	-	72	(8)-(12)
(14) (S/N)$_{REQUIRED}$	-	12	-	12	Table 2
(15) Margin	-	35	-	60	(13)-(14)
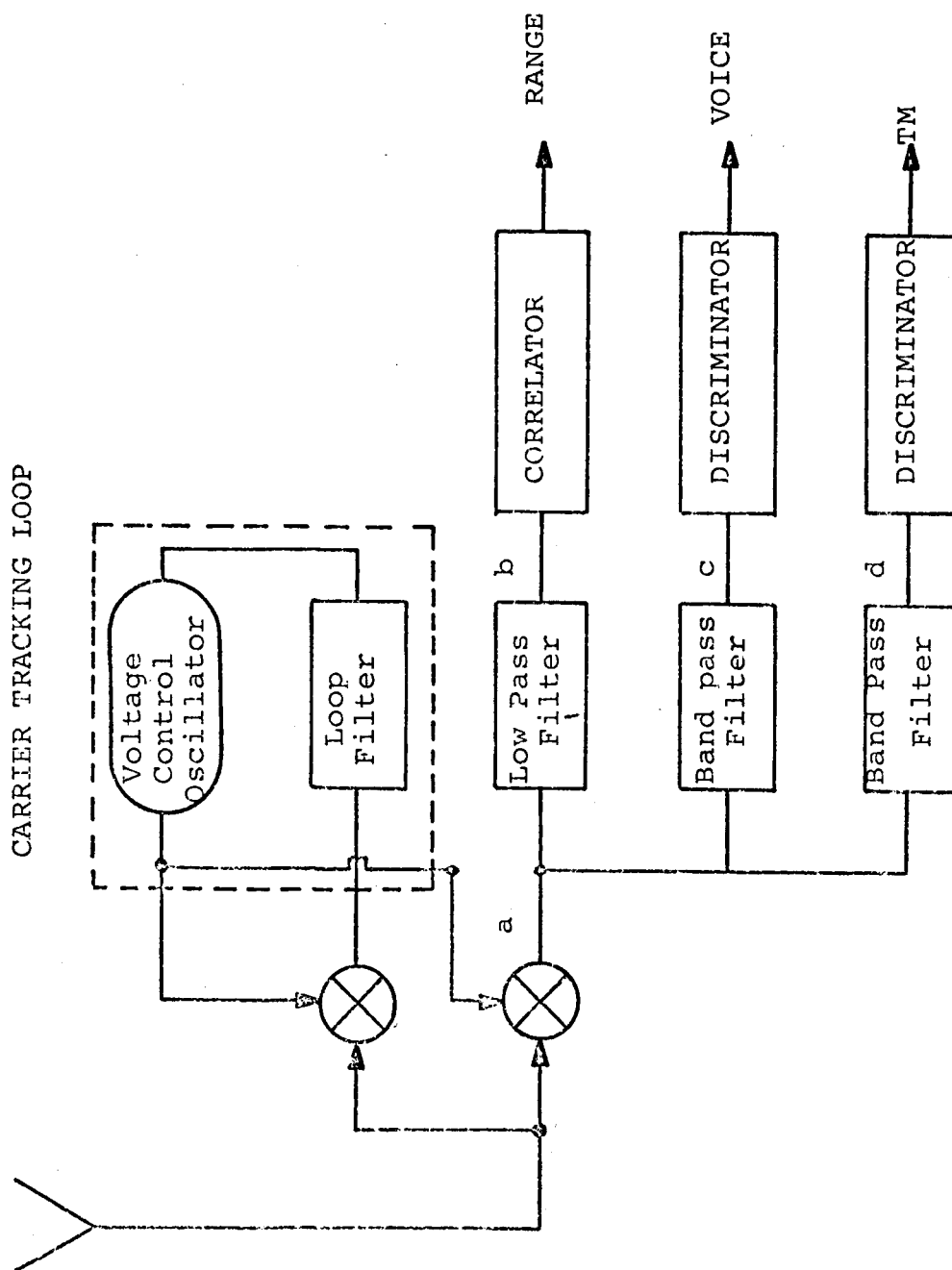
Figure 2
GROUND STATION RECEIVER

filters are then used to separate the sub-carriers prior to detection.

It can be shown that the effective signal to noise ratio at points a, b, c, and d of Figure 2 can be approximated by

$$(S/N)_a = \frac{A_c^2 \, J_0^2 \, (\beta_1) \, J_0^2 \, (\beta_2) \, \cos^2\beta_0}{2KTB_a} \tag{5}$$

$$(S/N)_b = \frac{A_c^2 \, J_0^2 \, (\beta_1) \, J_0^2 \, (\beta_2) \, \sin^2\beta_0}{2KTB_b}$$

$$(S/N)_c = \frac{A_c^2 \, J_1^2 \, (\beta_1) \, J_0^2 \, (\beta_2) \, \cos^2\beta_0}{2KTB_c}$$

$$(S/N)_d = \frac{A_c^2 \, J_0^2 \, (\beta_1) \, J_1^2(\beta_2) \, \cos^2\beta_0}{2KTB_d}$$

where A_c is the carrier amplitude, β_0, β_1, β_2 are the modulation indices of the ranging voice and telemetry channels, K is Boltzman's constant, T is the effective system noise temperature, and B_a, B_b, B_c, B_d are the equivalent noise bandwidths at points a, b, c, and d respectively.

These relationships were used to develop the system of equations described previously. The computed parameters of the equations are listed in Table 4. The equations are

$$35 + 10 \log_{10} \tfrac{1}{2}[\cos \beta_0 J_0(\beta_1) J_0(\beta_2)]^2 = A_0 \tag{6}$$

$$45 + 10 \log_{10} \tfrac{1}{2}[\sin \beta_0 J_0(\beta_1) J_0(\beta_2)]^2 = A_1$$

$$22 + 10 \log_{10} \tfrac{1}{2}[\cos \beta_0 J_1(\beta_1) J_0(\beta_2)]^2 = A_2$$

$$-B_{TM} + 56 + 10 \log_{10} \tfrac{1}{2}[\cos \beta_0 J_0(\beta_1) J_1(\beta_2)]^2 = A_3$$

The variables are the modulation indices (β_0, β_1, β_2), and the telemetry equivalent noise bandwidth (B_{TM}) which determines the information rate.

An additional constraint on the system of equations is that the sum of the modulation indices must be less than 2.4, because the type of carrier tracking loop used in this system will not function properly with a peak phase deviation greater than the first zero of the zero order Bessel function (Gardner and Kent, 1966).

A computer program was written in the MAD language for the IBM7094 to optimize the general system of equations. The form of the P_n and the values of A and B_n are left as inputs to be specified

TABLE 4

SYSTEM PARAMETERS FOR ALL CHANNELS

	Units	db	Source
All Channels			
(1) Received Signal Power		-140	Table 3
(2) Noise Spectral Density		-215	Table 3
(3) A		75	(1)-(2)
Carrier Tracking Loop			
Bandwidth = B_2	700 Hz	28	Table 2
$(S/N)_{REQUIRED}$		12	Table 2

$$75-28-12-10\log_{10}P_1=M_1$$

	Units	db	Source
Ranging Channel			
Bandwidth = B_2	0	1	Table 2
$(S/N)_{REQUIRED}$		32	Table 2

$$75-0-32-10\log_{10}P_2=M_2$$

	Units	db	Source
Voice Channel			
Bandwidth = B_3	20kc	43	Table 2
$(S/N)_{REQUIRED}$		10	Table 2

$$75-43-10-10\log_{10}P_3=M_3$$

	Units	db	Source
Telemetry Channel			
Bandwidth = B_4			Table 2
$(S/N)_{REQUIRED}$		19	Table 2

$$75-B-19-10\log_{10}P_3=M_4$$

at the time of execution. An abbreviated flow chart of the program applied to this specific system is shown in Figure 3. The program is used to select the optimum values of the modulation index and the maximum information rate and to calculate the performance of the channels. The results of the computer program are presented in Table 5. They indicate that all channels have positive performance margins, although based upon present requirements for the Apollo program the telemetry rate is too low to support the transmission of both scientific and operational data.

TABLE 5

CHARACTERISTICS OF THE OPTIMUM SPACECRAFT TO GROUND LINKS

Function	Modulation Index	Performance Margin db
Carrier	--	23.0
Ranging	0.6	11.1
Telemetry at 16 Kilobits per second	1.7	0.1
Voice	0.1	10.4

The same signal design, transmitter and receiver configuration can be used for the up-link, as this is the stronger of the two and will function satisfactorily if the down-link functions satisfactorily.

IV. An All Digital Spacecraft to Ground Link

For a channel which is peak power limited but not bandwidth limited, the proven optimum means of communications is simplex signaling. For the binary case this theoretical optimum is most nearly approached in practical systems by phase shift keying (PSK) with coherent demodulation and matched filter detection.

For the above PSK system the probability of bit error as a function of received power and the noise spectral density of the receiver is given by

$$P_e^b = \frac{1}{2}(1 - erf\sqrt{ST/2\eta}) \tag{7}$$

where

S = Received signal power in watts
T = Bit length in sec

```
        ┌─────────────┐
        │    READ     │
        │  C(1)..C(4) │
        └─────────────┘
               ↓
        ╭─────────────╮
        │ AMIN=-100   │
        ╰─────────────╯
               ↓
        ╱───────────────╲
        │      L1        │
        │ JA=.1,.1,2.4   │
        ╲───────────────╱
               ↓
        ╱───────────────╲
        │      L2        │
        │ JB=.1,.1,2.4   │
        ╲───────────────╱
               ↓
        ╱───────────────╲
        │      L3        │
        │ JC=.1,.1,2.4   │
        ╲───────────────╱
               ↓
     ╭─────────────────────╮  yes
     │ (JA+JB+JC) ≥ 2.4    │────→
     ╰─────────────────────╯
```

$$A_1 = C_1 + 10\text{Log}_{10}\left[J_0(JA)\,J_0(JB)\,\text{COS}(JC)\right]^2$$

$$A_2 = C_2 + 10\text{Log}_{10}\left[J_0(JA)\,J_0(JB)\,\text{SIN}(JC)\right]^2$$

$$A_3 = C_3 + 10\text{Log}_{10}\left[J_1(JA)\,J_0(JB)\,\text{COS}(JC)\right]^2$$

$$A_4 = C_4 + 10\text{Log}_{10}\left[J_0(JA)\,J_1(JB)\,\text{COS}(JC)\right]^2$$

```
     ╭──────────────────────────────╮  no
     │ MIN[A_1,A_2,A_3,A_4] > AMIN   │────→
     ╰──────────────────────────────╯
```

$$A_1 = M_1,\ A_2 = M_2,\ A_3 = M_3,\ A_4 = M_4,\ \text{AMIN} = \text{MIN}\ A_1,A_2,A_3,A_4$$
$$J_1 = JA,\ J_2 = JB,\ J_3 = JC$$

```
        ┌─────────────┐
        │    PRINT     │
        │ M_1,M_2,M_3,M_4 │
        │ J_1,J_2,J_3  │
        └─────────────┘
```
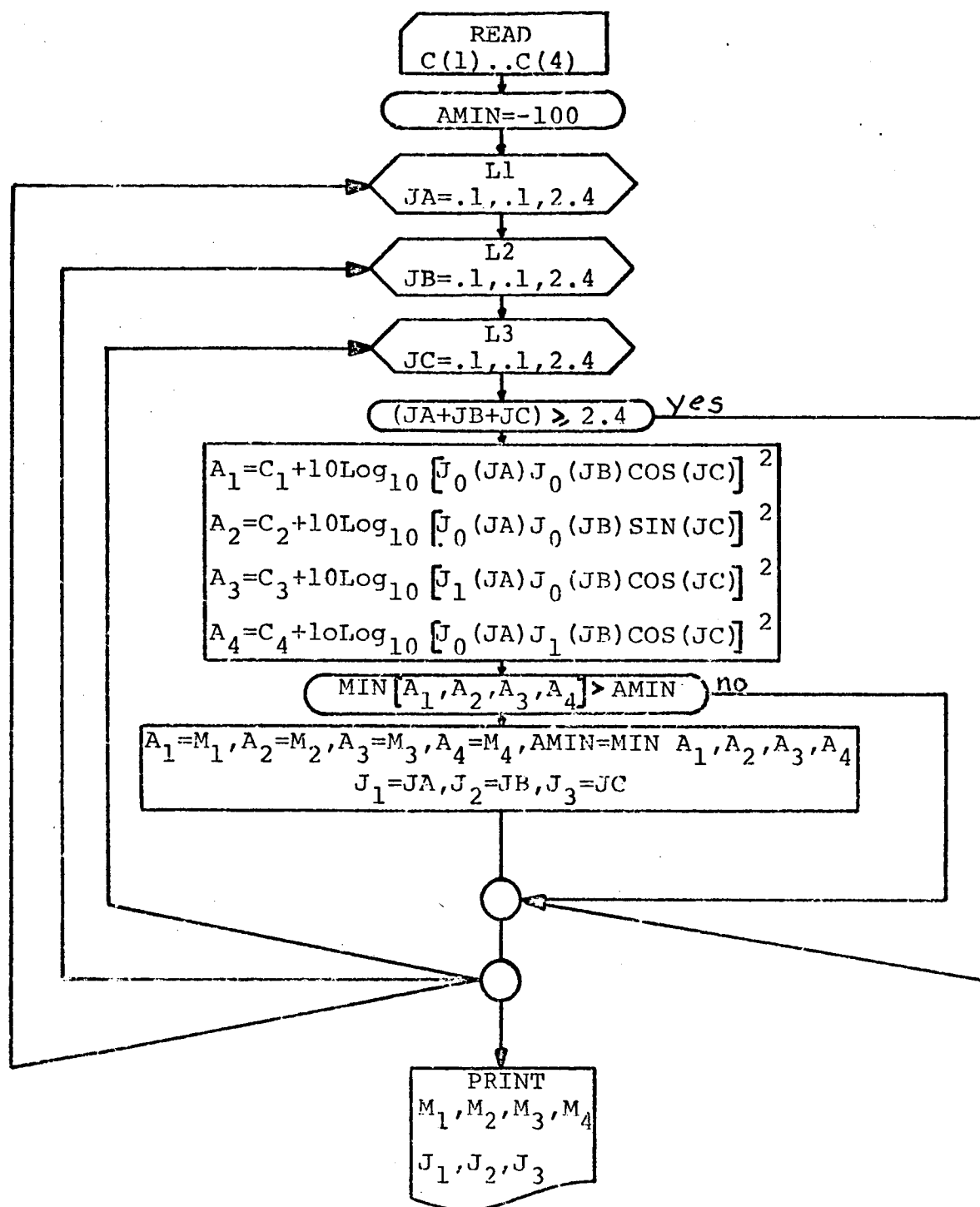
Figure 3

η = Spectral density of the input noise in watts/cps
and erf is the so-called error function.

$$\text{erf } x = \frac{2}{\pi} \int_0^x e^{-y^2/2} \, dy \tag{8}$$

The quantity $P_e{}^b$ is plotted in Figure 4 versus $ST/2\eta$.

For most data transmission applications the minimum acceptable bit error rate is

$$P_e{}^b = 10^{-3} \tag{9}$$

It can be seen from Figure 4 that this corresponds to

$$ST/2\eta = 6.8 \text{ db} \tag{10}$$

Furthermore Table 3 gives

$$S/2\eta = -140 - (-215) = 75 \text{ db} \tag{11}$$

for the spacecraft-to-ground communications link operating at interplanetary distances. It follows that the theoretical maximum data rate, $R = \frac{1}{T}$, for this system is

$$R = 68.2 \text{ db} \tag{12}$$

$$= 6.61 \, (10)^6 \text{ BPS}$$

The maximum data rate for the optimum conventional system of Chapter III would be $16(10)^3$BPS for the telemetry plus the voice which could be digitized to form a $1.2(10)^3$BPS bit stream (Fillipowsky and Meuhldorf, 1965). This would indicate a superiority of 375:1 for the digital system over the conventional system, when both figures ignore the contribution of the PRN ranging code, which is neither clearly understood nor discussed in the literature.

Two major problems must be solved before full advantage can be taken of this high theoretical data rate. First a method must be developed so that all the information can be combined into a single digital waveform in such a way that the spacecraft transponder does not have to process the pseudo-random ranging code. Second a means of obtaining a highly stable coherent reference must be obtained. Only the second of these two problems will be discussed here.
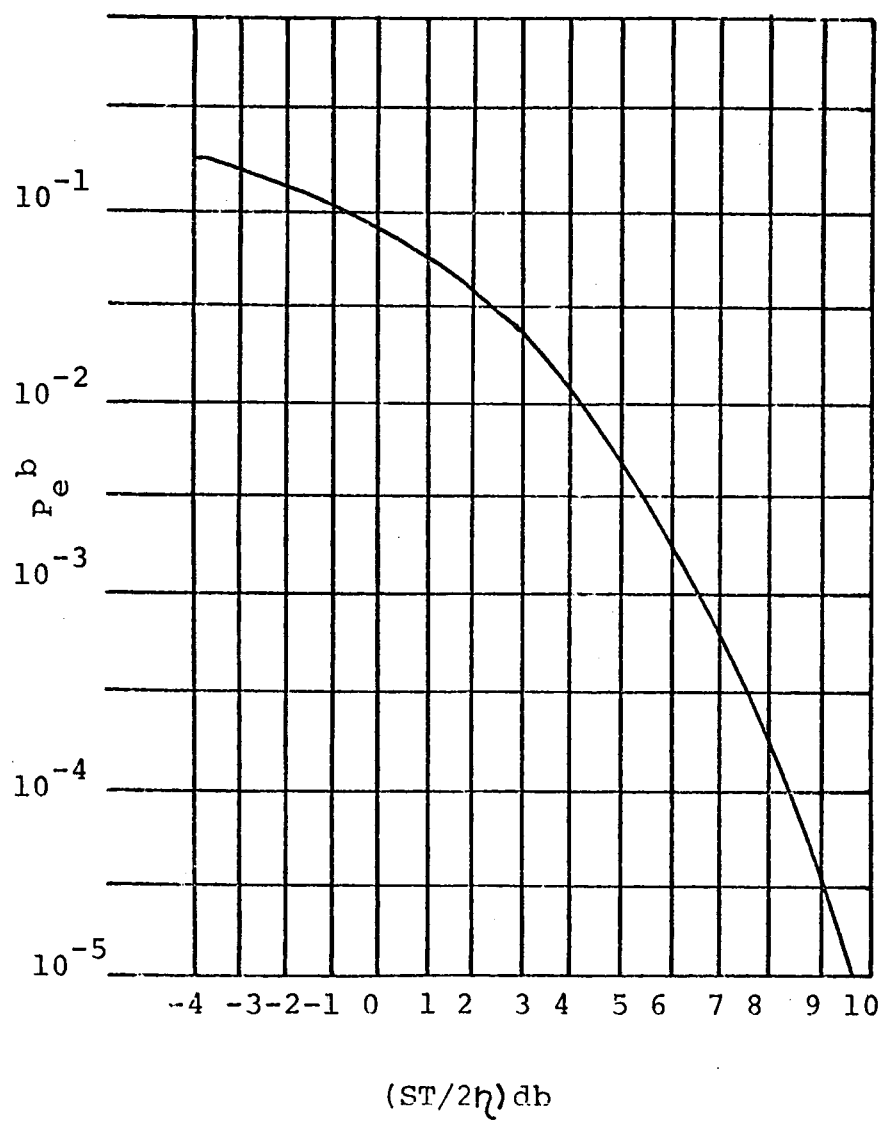
Figure 4

PERFORMANCE OF A PSK SYSTEM

A commonly used method for providing synchronization, in most PSK systems, is to transmit a discrete synchronization signal. Such a technique is often described as auxiliary channel synchronization.

In peak power limited systems, auxiliary channel synchronization decreases the amount of energy available for transmission. For example one commonly used technique is to transmit

$$S(t) = \cos(\omega_c t + \beta m(t)) \tag{13}$$

where $\quad o \leq \beta \leq \pi$

or $\quad S(t) = \cos\beta \cos\omega_c t - m(t) \sin\beta \sin\omega_c t$

From Equation (13) it can be seen that $\cos^2\beta$ of the available energy is used to generate a spectral component at ω_c which can be tracked by a phase lock loop to provide a coherent reference. The energy in the auxiliary channel is obviously provided at the expense of energy in the information channel. The division of energy between the two channels is shown in Figure 5. Therefore the desirability to put as much energy in the information channel as possible is in conflict with the requirement to make β as small as possible in order to provide adequate synchronization.

It has also been shown (Lindsey, 1965) that in auxiliary systems the auxiliary signal is often disturbed by the channel in a manner that is different from that of the information signal thus adversely effecting the performance.

As a result of the above it appears that a scheme which allows one to derive a reference from the information itself would be useful. One such scheme is the "carrier hatching loop," which is shown in Figure 6. The name comes from the fact that the band-pass filter is realized by use of a phase lock loop.

If as before:

$$s = A \cos[\omega_c t + m(t) \frac{\pi}{2}] \tag{14}$$

then:

$$s^2 = \frac{A^2}{2} + \frac{A^2}{2} \cos 2\omega_c t$$

If the band pass filter is centered at $2\omega_c$ then its output can be divided in frequency by two and used as a reference signal. The key to the performance of this technique is the signal to noise performance of the loop. The output signal to noise ratio can be shown to be

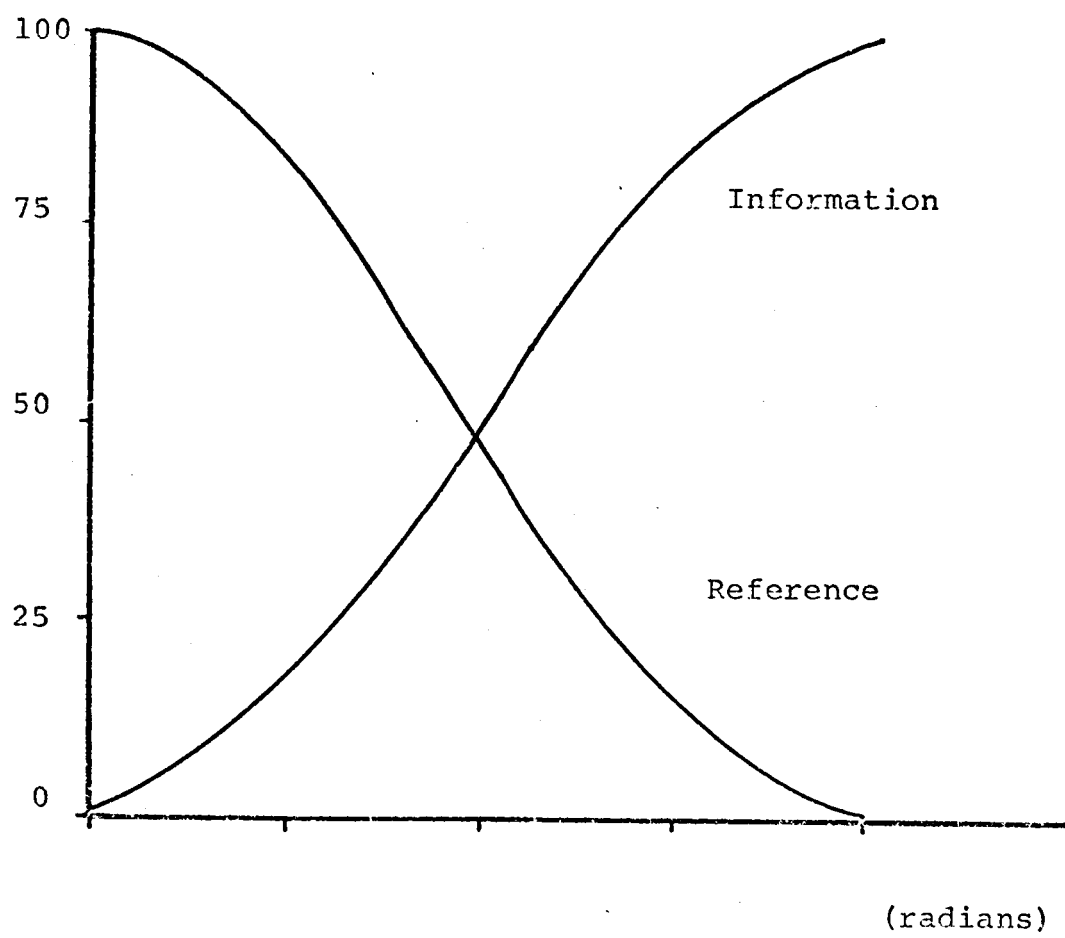$$(S/N)_{out} = \frac{\frac{A^2}{2} \pi^2}{2 \eta \omega_{BP}} \tag{15}$$

Figure 5

DIVISION OF TOTAL ENERGY VERSUS β
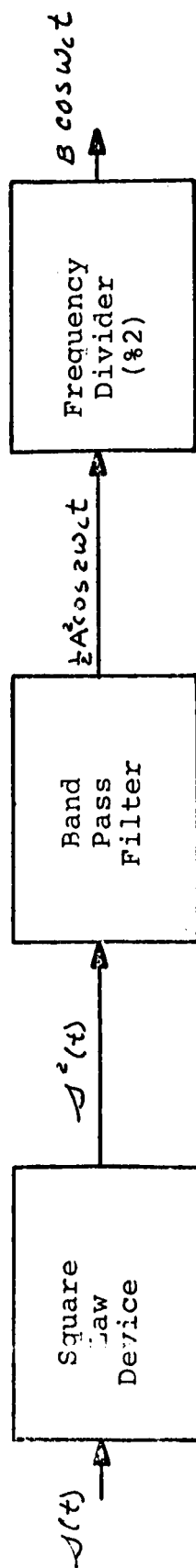
Figure 6
A Carrier Hatching Loop

where ωBP is the radian bandwidth of the filter. For a typical value for a phase locked tracking filter S/2η = 75 db and ωp = 2π(700) (Bunce, 1965), the output signal-to-noise ratio is 69 db. This signal-to-noise ratio would be adequate to derive a coherent reference with negligible phase error.

In addition to providing a coherent reference for the demodulation of the data the reference can also be used to coherently demodulate the angle tracking information. This utilization obviates the need for transmitting a wasteful reference for angle tracking as is presently done in many applications.

A functional block diagram of the ground receiver for the PSK system is shown in Figure 7.

V. Conclusions

While it is too early to predict exactly what communications systems will be used for the manned interplanetary missions, it appears that the all-digital PSK system is worthy of considerations. The system displays considerable theoretical advantage over the present unified S-band system, is capable of integration into the MSFN, and its few remaining unresolved problems are presently under intense study by many orgianizations.
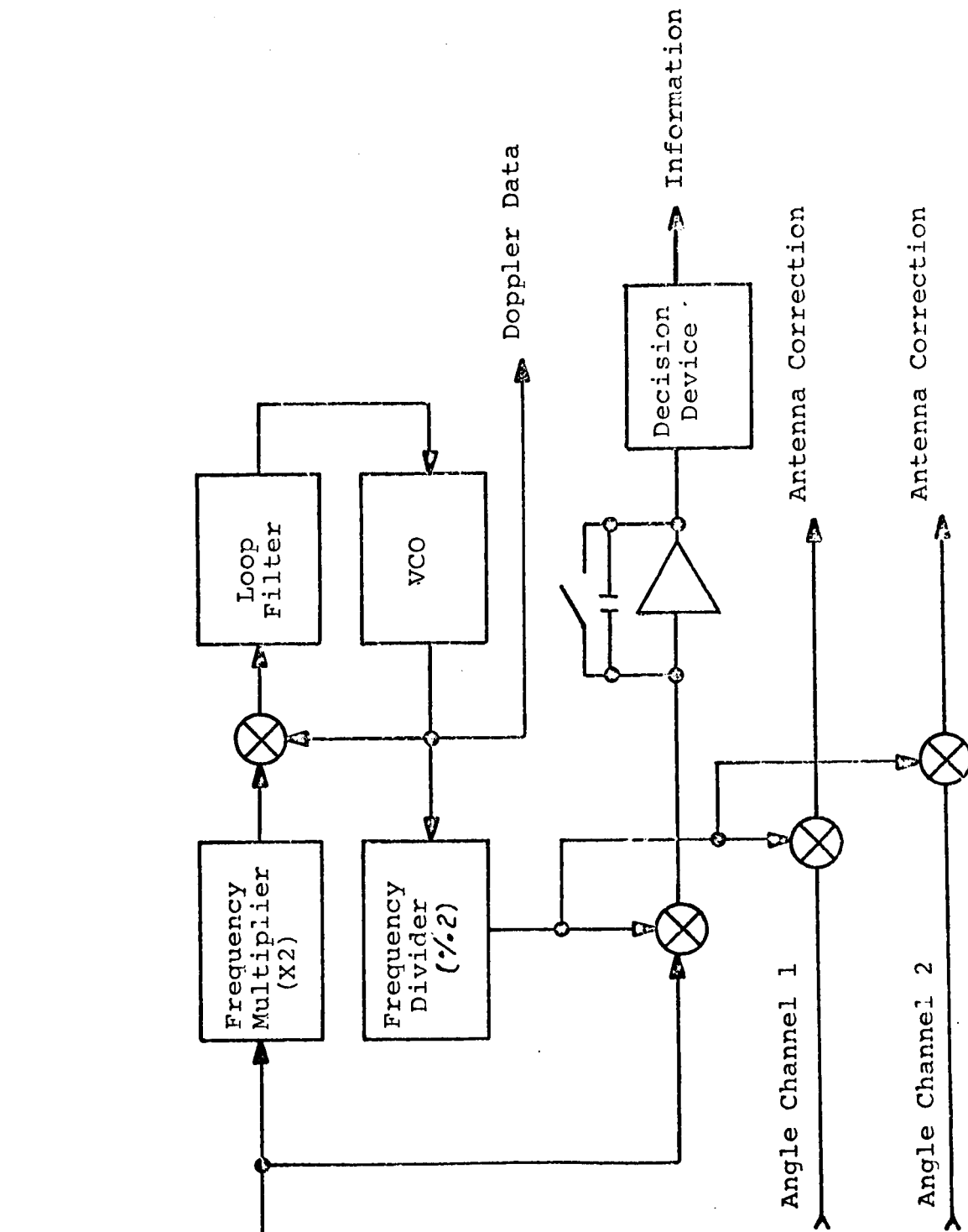
Figure 7
A PSK GROUND RECEIVING SYSTEM

BIBLIOGRAPHY

Books and Articles

Bunce, R. "Receiver Exciter Subsystem", Proceedings of the Apollo Unified S-Band Technical Conference--NASA SP 87, July 14, 1965.

Downing, J.J. Modulation Systems and Noise. Englewood Cliffs, N.J.: Prentice-Hall, Inc., (1964).

Easterling, M., and R. Goldstein. "The Interplanetary Medium and S-Band Telecommunications" Jet Propulsion Laboratory Technical Report No. 32-825, September 1, 1965.

Filipowsky, R.E., and E.I. Muehldorf. Space Communications Systems. Englewood Cliffs, N.J.: Prentice-Hall, Inc., (1965).

Gardner, R.M., and S. Kent. Theory of Phaselock Techniques (to be published).

Golomb, B. et. al. Digital Communications. Englewood Cliffs, N.J.: Prentice-Hall, Inc., (1964).

Grimm, H.H. "Fundamental Limitations of External Noise," IRE Transactions on Instrumentation. December 1959, pp. 97-103.

Hancock, J.C., and J.A. Wintz. Signal Detection Theory. New York: McGraw-Hill Book Company, (1966).

Oliver, B.M. "Thermal and Quantum Noise", Proc. of the I.E.E.E., May 1965, pp. 436-454.

Painter, J.H. and G. Hondros. Unified S-Band Telecommunication Techniques for Apollo, Volume I. National Aeronautics and Space Administration, Manned Spacecraft Center, Houston, Texas, 1965.

Springett, J.C. "Telemetry and Command Techniques for Planetary Spacecraft" Advances in Communications Systems, Volume I, New York: The Academic Press, 1965.

Tabor, W.L., and J.T. Sibilia. "Masers for the Telstar Satellite Communications Experiment", Bell Syst. Tech. Journal, Volume 142, July 1963, pp. 1863-1866.

Viktor, W.K., R. Titsworth and E. Rechtin. "Telecommun-
ication Aspects of a Manned Mars Mission" Jet Propul-
sion Laboratory Technical Report No. 32-501, August
20, 1963.

Watson, G.N. Theory of Bessel Functions. New York: The
MacMillan Company, (1944).

Wozencraft, J.M. and I.W. Jacobs. Principles of Communication
Engineering. New York: John Wiley and Sons, Inc.,
(1965).

Reports

"GFE Command and Service Module Manned Space Flight Network
Signal Performance and Interface Specification, Block II"
North American Aviation Inc., Document No. SID 64-1613,
February 22, 1965.

"Manned Mars and/or Venus Flyby Vehicle Systems Study". North
American Aviation Inc., Document No. SID 65-761-3A,
July, 1965.

"S-Band Circuit Margin Interface Control Document, Block II".
North American Aviation Inc., Document No. MH01-13001-414,
July 30, 1965.